Analytical Study of the Mercator Map Factor using Fourier Series

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1. Introduction

In the semi-implicit scheme of the ALADIN Model, the value of the map factor is a constant equal to its maximum value over the integration domain. This simplification seems to be legitimate for limited area models, in which the map factor remains close to the unity. However, its values are greater than the unity when using large domains. Thus, in this case and from the stability point of view, the variability of this factor over the integration domain should be considered.

Yessad and Bénard [6] make clear how to solve this problem at the ARPEGE Global Model by writing the map factor as a combination of Legendre polynomials of orders zero and one. In this case, the variations of the map factor are exactly and entirely included in the first component of the spectral formulation. The solution of the Helmholtz equation yielded by the model consists of the inversion of a pentadiagonal matrix for each zonal number and time-step.

In order to apply the same method to the ALADIN Model, the formulation of the map factor in this model should have a simple pattern at the spectral space: linear combination of low-order Fourier harmonics. Consequently, the mapping factor dependency in the spectral part of the computations can be included by a simple extra multi-diagonal operator multiplication. This expression should improve the behaviour of the model when increasing the integration domain and, at the same time, the extra computation and memory cost should be small. An inconvenient of using this method is the requisite of knowing the mapping factor analytic form for the different projections in the bi-Fourier spectral space (Voitus, 2004) [5].

The aim of this article is to show the analytical expression of the Mercator map factor expanded as a Fourier series and study the behaviour of this Fourier approximation, in order to be included in the ALADIN

Model. First, as starting point of this study, the general expression of this map factor is presented. Second, the authors have expanded this expression as a Fourier series and calculated its coefficients analytically. In this way, the inconvenient explained previously is removed, allowing their implementation in the ALADIN Model for the rotated Mercator projection. Third, several comparative studies are showed with different size of the integration domain. Finally, a summary of these points is given.

2. The Mercator Map Factor

For a Mercator projection, the expression of its map factor m in spherical coordinates is:

$$\mathrm{m}(\lambda, \varphi) = \frac{1}{\cos(\varphi)},$$

where φ is the latitude and λ is the longitude. In cartesian coordinates (x, y) of the plane:

$$\mathrm{m}(x, y) = \mathrm{m}(y) = \cosh(y/a),$$

where a is the radius of the Earth and $y \in [-\frac{L_y}{2}, \frac{L_y}{2}]$. We define L_y as the product of a factor f by the radius of the Earth, $a \simeq 6371$km.

As it can be seen, this map factor depends on the latitude only.

Besides, let us say that the choice of the Mercator projection has been made on purpose, as its map factor expression (hyperbolic cosine) is one of the best suited for developing a Fourier cosine series.

3. Fourier Coefficients of the Mercator Map Factor

Taking into account that the term which appears at the discrete equations of the model is the square of the map factor m^2, the Fourier series coefficients calculated are the first ones of this even function belonging to $L^2[-\frac{L_y}{2}, \frac{L_y}{2}]$; where $[-\frac{L_y}{2}, \frac{L_y}{2}]$ is the domain considered. It is known that for any even function of L^2 in a bounded interval there is a Fourier cosine series converging to the function in that interval. As it has been already said, m^2 fulfills these conditions. Consequently, there is a Fourier cosine series converging to m^2 in $L^2[-\frac{L_y}{2}, \frac{L_y}{2}]$. To be precise,

$$\mathrm{m}^2(y) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{2\pi n x}{L_y} y\Big)\, dy, \tag{1}$$

with

$$a_n = \frac{4}{L_y} \int_0^{\frac{L_y}{2}} \mathrm{m}^2(y) \cos\Big(\frac{2\pi n x}{L_y} y\Big)\, dy, \quad n = 0, 1, 2, ... \tag{2}$$

These coefficients have been computated analytically by the authors. Because of space, the detailed development of this calculation is included in the Appendix. Its analytical computation provides the following values for the Fourier coefficients depending on the parameter f:

$$a_0 = \frac{1}{2\mathrm{f}} \left(e^{\mathrm{f}} - e^{-\mathrm{f}}\right) + 1 \tag{3}$$

$$a_n = \frac{(-1)^n \mathrm{f}}{2[(\pi n)^2 + \mathrm{f}^2]} \left(e^{\mathrm{f}} - e^{-\mathrm{f}}\right), \quad n = 1, 2, ... \tag{4}$$

4. Numerical Tests

In this section two types of test are carried out. The first one shows the behaviour of the finite Fourier series of m^2 using different number of coefficients. The second one analyzes the effect of the integration domain size considering the same Fourier truncation with three coefficients.

The first one consists in comparing two aspects:

- a) the Fourier cosine series of m^2, for the same domain, respect to its exact value for three different truncations, in order to study the effect of the number of coefficients and
- b) the adjustment of each Fourier truncation with the approximation of m^2 by its maximum value in $[-\frac{L_y}{2}, \frac{L_y}{2}]$.

In the second test, the previous comparisons for the truncation with three coefficients are repeated with four different domains L_y.

The mathematical tool used to compare the distinct functions is L^2-norm. As m^2 and its approximations belong to $L^2[-\frac{L_y}{2}, \frac{L_y}{2}]$, L^2-norm of the difference between the exact value of m^2 and its approximations is calculated by the following expression:

$$||\, m^2 - approx|| = \Big(\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} |m^2(y) - approx(y)|^2\, dy \Big)^{\frac{1}{2}},$$

obtaining an index which measures the distance between both functions.

Qualitative Results

Figures 1.a, 2.a and 3.a show the behaviour of the square of the exact value of the map factor m^2 through the domain versus the Fourier truncations (denoted by $\hat{m}^2$). From these figures, it can be seen that the more coefficients of the Fourier series of $\hat{m}^2$ are considered, the closer to the exact value of m^2 the truncations are. Besides, it is remarkable the fact that the values of the map factor increase towards the boundaries of the integration domain, being the unity in the centre. On the other hand, figures 1.b, 2.b and 3.b display the considerable and progressive improvement using the Fourier approximations to m^2 instead of the approximation given by the maximum value in the domain, m_*^2 . The size of the integration domain has been normalized so that in the following tests we can compare results with different values of L_y.

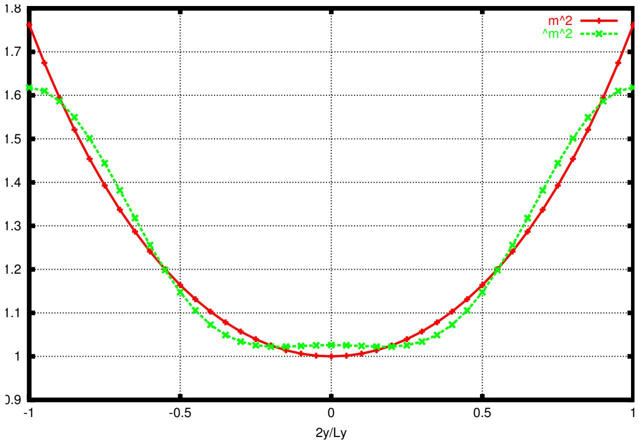

Figure 1.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1 and a_2 in green. Ly=10050km

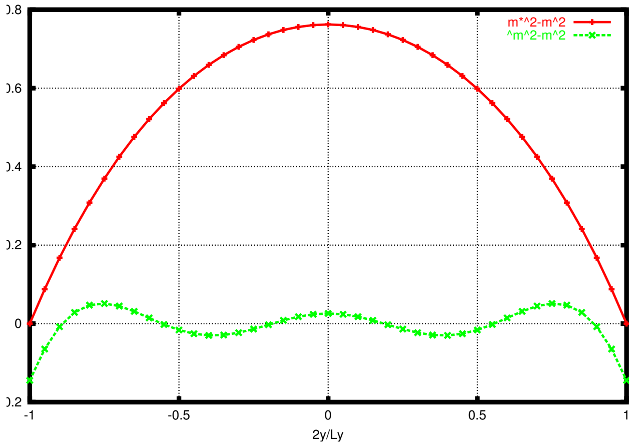

Figure 1.b: Comparison of the Fourier approximation using the coefficients a_0, a_1 y a_2, in green, and the maximum approximation, in red, to the exact value. Ly=10050km

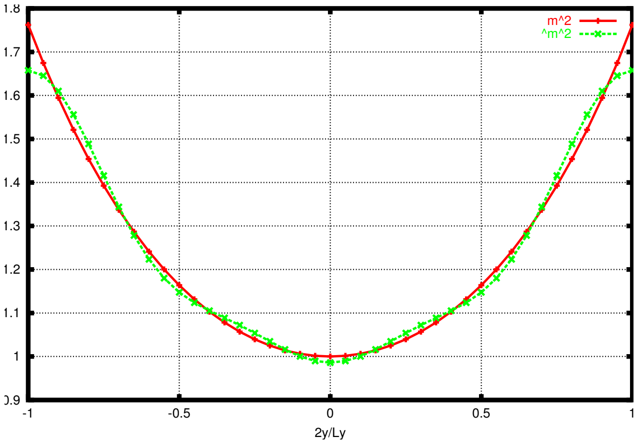

Figure 2.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1, a_2 and a_3 in green. Ly=10050km

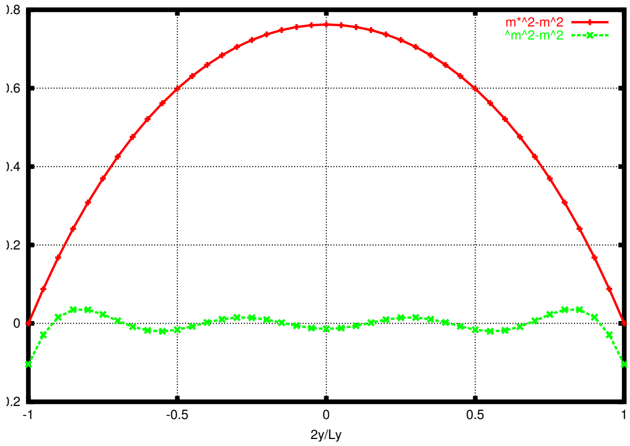

Figure 2.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1, a_2 and a_3, in green, and the maximum approximation, in red, to the exact value. Ly=10050km

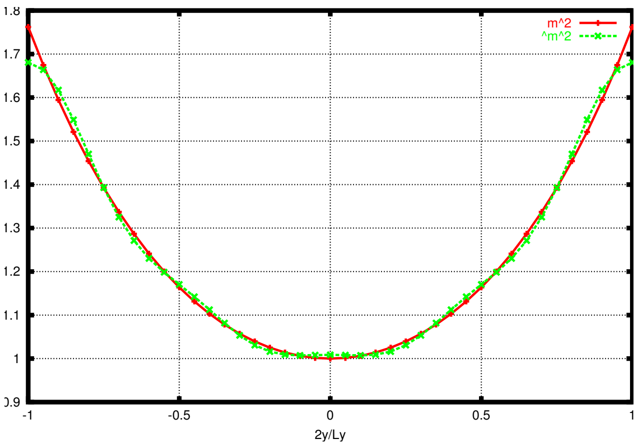

Figure 3.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1, a_2, a_3 and a_4 in green. Ly=10050km

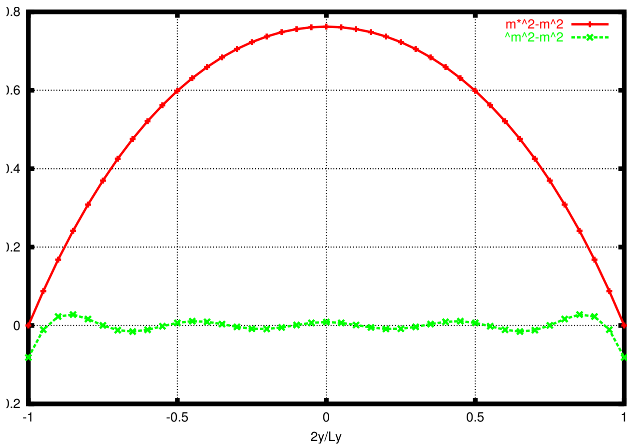

Figure 3.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1, a_2, a_3 and a_4, in green, and the maximum approximation, in red, to the exact value. Ly=10050km

Next, we examine the effect of the size of the integration domain in both approximations respect to the exact value of m^2. To do this, four different sizes are considered in the 3-coefficient Fourier approximation: $L_y = 8375$km (f $= 1.3146$), $L_y = 6700$km (f $= 1.0516$), $L_y = 5025$km (f $= 0.7887$) and $L_y = 2512.5$km (f $= 0.3944$).

Having a look at the graphs of m^2 in figures 1.a, 4.a, 5.a, 6.a and 7.a it is easily noticed that the hyperbolic cosine always gets the unity in the middle of the interval, while its values towards the domain boundaries tend to decrease when the size of the interval gets smaller. Hence, observing figures 1.b, 4.b, 5.b, 6.b and 7.b, it can be concluded that the greater the domain under study is, the better the Fourier truncation approximation of the map factor is in comparison with the maximum value approximation. However, when the size of the integration domain is smaller than 5000 km, the maximum value over the area of the map factor is a suitable approximation (figures 7.a and 7.b).

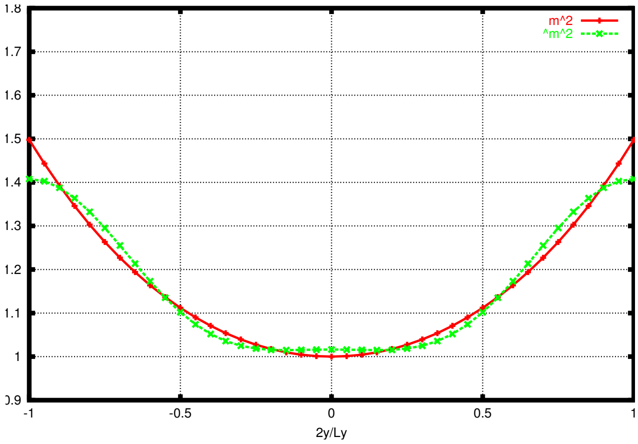

Figure 4.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1 and a_2 in green. Ly=8375km

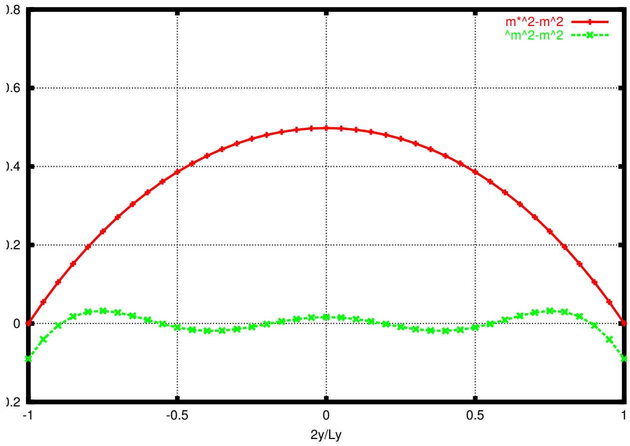

Figure 4.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1 and a_2, in green, and the maximum approximation, in red, to the exact value. Ly=8375km

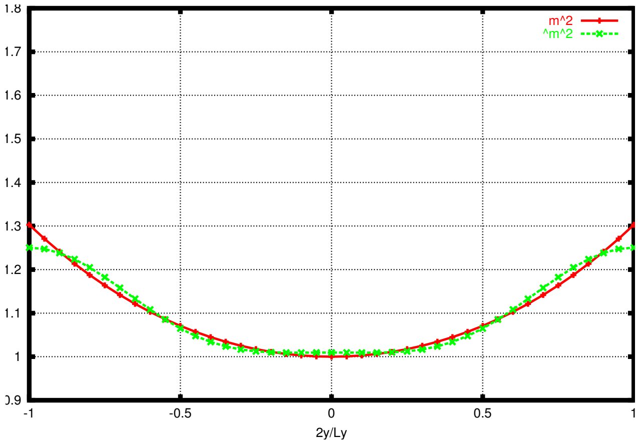

Figure 5.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1 and a_2 in green. Ly=6700km

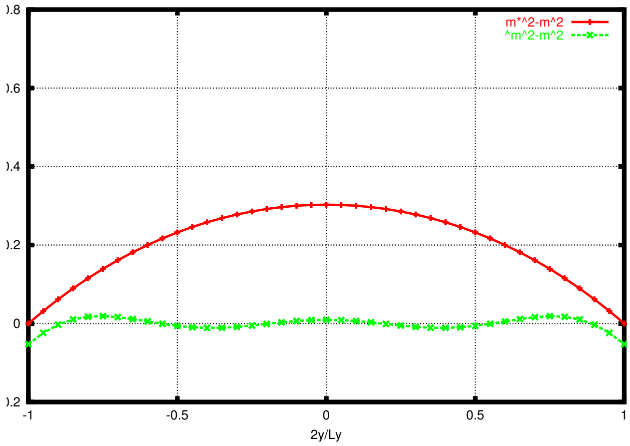

Figure 5.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1 and a_2, in green, and the maximum approximation, in red, to the exact value. Ly=6700km

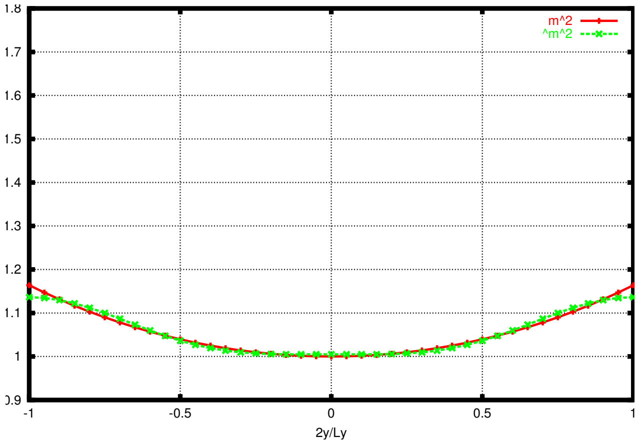

Figure 6.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1 and a_2 in green. Ly=5025km

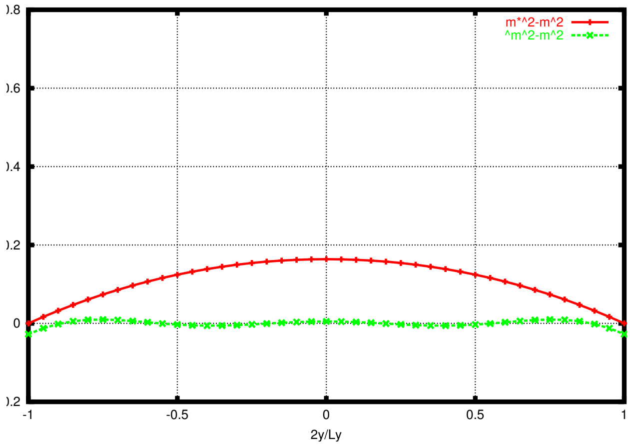

Figure 6.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1 and a_2, in green, and the maximum approximation, in red, to the exact value. Ly=5025km

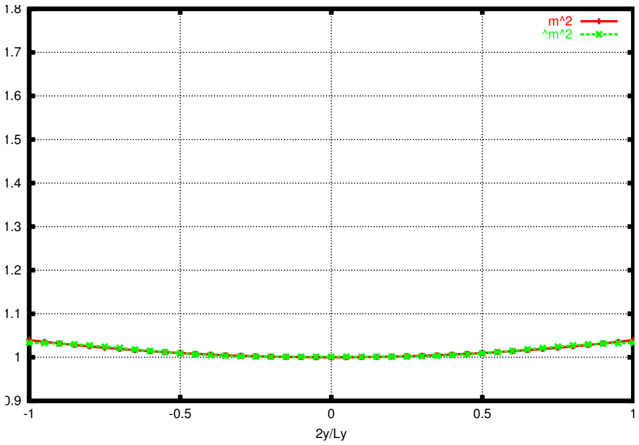

Figure 7.a: The square of the exact value of the map factor in red and the square of the approximate value with the coefficients: a_0, a_1 and a_2 in green. Ly=2512.5km

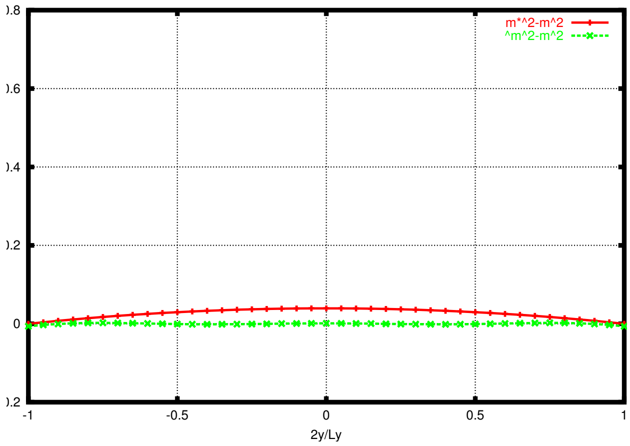

Figure 7.b: Comparison of the Fourier approximation using the coefficients:a_0, a_1 and a_2, in green, and the maximum approximation, in red, to the exact value. Ly=2512.5km

Quantitative Results

Tables 1 and 2 show the values of L2-norm (normalized and no-normalized, respectively) of the functions $m^2 - m_*^2$ and $m^2 - \hat{m}^2$ over the interval $[-\frac{L_y}{2}, \frac{L_y}{2}]$ for four sizes of integration domain L_y. m_*^2 denotes the maximum value of m^2 over the area and $\hat{m}^2$ the Fourier approximations of the map factor (with three different truncations). These values constitute indexes which providing a quantitative measure of the distance between the functions compared each other, m^2 with m_*^2 and m^2 with $\hat{m}^2$. Therefore, conclusions about the behavior of the different approximations of the hyperbolic cosine to the exact values can be obtained and supported by numerical results.

It can be seen that when the integration domain is small, L2-norm is small as well. But, when the former is greater than 5000 km, the no-normalized L2-norm using the maximum value approximation takes high values, whereas the no-normalized L2-norm of the Fourier approximation is one or even two order of magnitude smaller, indicating a much better behavior . A similar conclusion can be obtained by looking at the values of the normalized L2-norm, which gives a proportional value of the norm depending on the size of the integration domain.

L_y		10050 km	5025 km	2512.5 km	1256.25 km
$\|\|m^2 - m_*^2\|\|_{L_2}$		$8.089146 \cdot 10^{-1}$	$1.703546 \cdot 10^{-1}$	$4.075417 \cdot 10^{-2}$	$1.007624 \cdot 10^{-2}$
$\|\|m^2 - \hat{m}^2\|\|_{L_2}$	a_0, a_1, a_2	$5.314936 \cdot 10^{-2}$	$1.016265 \cdot 10^{-2}$	$2.364249 \cdot 10^{-3}$	$5.803171 \cdot 10^{-4}$
	a_0, a_1, a_2, a_3	$3.405282 \cdot 10^{-2}$	$6.463488 \cdot 10^{-3}$	$1.500847 \cdot 10^{-3}$	$3.682166 \cdot 10^{-4}$
	a_0, a_1, a_2, a_3, a_4	$2.458599 \cdot 10^{-2}$	$4.651628 \cdot 10^{-3}$	$1.079247 \cdot 10^{-3}$	$2.647274 \cdot 10^{-4}$

Table 1: normalized L2-Norm

L_y		10050 km	5025 km	2512.5 km	1256.25 km
$\|\|m^2 - m_*^2\|\|_{L_2}$		$1.813305 \cdot 10^{3}$	$2.700269 \cdot 10^{2}$	$4.567832 \cdot 10$	7.985862
$\|\|m^2 - \hat{m}^2\|\|_{L_2}$	a_0, a_1, a_2	$1.144453 \cdot 10^{2}$	$1.548690 \cdot 10$	2.548175	$4.422928 \cdot 10^{-1}$
	a_0, a_1, a_2, a_3	$7.099142 \cdot 10$	9.535758	1.566021	$2.716886 \cdot 10^{-1}$
	a_0, a_1, a_2, a_3, a_4	$4.927779 \cdot 10$	6.597384	1.082561	$1.877740 \cdot 10^{-1}$

Table 2: No-normalized L2-Norm

Summary

The main conclusions of this study can be summarized in the following points:

- The variability of the Mercator Map Factor should be considered for large domains.
- The analytical formula for the coefficients of the Fourier series of the rotated Mercator map factor has been obtained.
- The difference between using the Fourier approximation and the maximum value respect to the exact value of the map factor increases with the size of the integration domain.
- If the size of the integration domain is greater than 5000 km, the adjustment of the map factor by the first three coefficients of the Fourier series is much better than its maximum value over the area.
- The Fourier approximation using three coefficients is accurate enough for our purpose. Consequently, these coefficients can be implemented in the Helmholtz equation of the ALADIN Model.

References

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Appendix

The general expression of the Fourier coefficients of the map factor in the cosine basis

$$\cos\Big(\frac{2\pi nx}{L_y}y\Big), \quad n = 0, 1, 2, ...$$

is given by the formula:

$$a_n = \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \mathrm{m}^2(y)\cos\Big(\frac{2\pi nx}{L_y}y\Big)\,dy, \quad n = 0, 1, 2, ...$$

where $L_y = \mathrm{f}a$.

In this appendix, we obtain a simple expression for the Fourier coefficients of the Mercator map factor.

- $\underline{n = 0}$

$$a_0 = \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \cosh^2(y)\,dy = \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \Big(\frac{e^{\frac{y}{a}} + e^{-\frac{y}{a}}}{2}\Big)^2 dy =$$

$$= \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \frac{e^{\frac{2y}{a}} + e^{-\frac{2y}{a}} + 2e^{\frac{y}{a}}e^{-\frac{y}{a}}}{4}dy = \frac{1}{L_y}\int_0^{\frac{L_y}{2}} (e^{\frac{2y}{a}} + e^{-\frac{2y}{a}} + 2)\,dy =$$

$$= \frac{1}{L_y}\Big[\frac{a}{2}e^{\frac{2y}{a}} - \frac{a}{2}e^{\frac{2y}{a}} + 2y\Big]_{y=0}^{y=\frac{L_y}{2}} = \frac{1}{L_y}\Big(\frac{a}{2}\Big(e^{\frac{L_y}{a}} - e^{-\frac{L_y}{a}}\Big) + L_y\Big).$$

Replacing L_y by $\mathrm{f}a$ we get,

$$a_0 = \frac{1}{\mathrm{f}a}\Big(\frac{a}{2}\Big(e^{\mathrm{f}} - e^{-\mathrm{f}}\Big) + \mathrm{f}a\Big).$$

Then

$$a_0 = \frac{1}{2\mathrm{f}}\Big(e^{\mathrm{f}} - e^{-\mathrm{f}}\Big) + 1.$$

- $\underline{n \geq 1}$

$$a_n = \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \cosh^2(y)\cos\Big(\frac{2\pi nx}{L_y}y\Big)\,dy = \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \Big(\frac{e^{\frac{y}{a}} + e^{-\frac{y}{a}}}{2}\Big)^2\cos\Big(\frac{2\pi nx}{L_y}y\Big)\,dy =$$

$$= \frac{4}{L_y}\int_0^{\frac{L_y}{2}} \frac{e^{\frac{2y}{a}} + e^{-\frac{2y}{a}} + 2e^{\frac{y}{a}}e^{-\frac{y}{a}}}{4}\cos\Big(\frac{2\pi nx}{L_y}y\Big)\,dy = \frac{1}{L_y}\int_0^{\frac{L_y}{2}} (e^{\frac{2y}{a}} + e^{-\frac{2y}{a}} + 2)\cos\Big(\frac{2\pi nx}{L_y}y\Big)\,dy. \tag{5}$$

This integral is divided in three integrals, corresponding to each one of the additions, respectively.

– (I)

$$(I) = \int_0^{\frac{L_y}{2}} e^{\frac{2y}{a}} \cos\Big(\frac{2\pi n x}{L_y} y\Big)\, dy.$$

Applying integration by parts:

$$(I) = \Big[\frac{L_y}{2\pi n}\, e^{\frac{2y}{a}} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big)\Big]_{y=0}^{y=\frac{L_y}{2}} - \int_0^{\frac{L_y}{2}} \frac{L_y}{2\pi n} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big)\, \frac{2}{a}\, e^{\frac{2y}{a}}\, dy =$$

$$= \Big[\frac{L_y}{2\pi n}\, e^{\frac{2y}{a}} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big)\Big]_{y=0}^{y=\frac{L_y}{2}} - \int_0^{\frac{L_y}{2}} \frac{\mathrm{f}}{\pi n}\, e^{\frac{2y}{a}} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big)\, dy.$$

Using integration by parts again:

$$(I) = \Big[\frac{L_y}{2\pi n}\, e^{\frac{2y}{a}} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big) + \frac{\mathrm{f} L_y}{2n^2\pi^2}\, e^{\frac{2y}{a}} \cos\Big(\frac{2n\pi}{L_y} y\Big)\Big]_{y=0}^{y=\frac{L_y}{2}} - \frac{\mathrm{f}^2}{(n\pi)^2} \int_0^{\frac{L_y}{2}} e^{\frac{2y}{a}} \cos\Big(\frac{2n\pi}{L_y} y\Big)\, dy =$$

$$= \Big[\frac{L_y}{2\pi n}\, e^{\frac{2y}{a}} \,\mathrm{sen}\Big(\frac{2\pi n}{L_y} y\Big) + \frac{\mathrm{f} L_y}{2n^2\pi^2}\, e^{\frac{2y}{a}} \cos\Big(\frac{2\pi L_y}{2n^2\pi^2}\, e^{\frac{2y}{a}} \cos\Big(\frac{2\pi n}{L_y} y\Big)\Big]_{y=0}^{y=\frac{L_y}{2}} \frac{\mathrm{f}^2}{(\pi n)^2} (I).$$

Then,

$$\Big[1 + \Big(\frac{\mathrm{f}}{\pi n}\Big)^2\Big] (I) = \frac{L_y}{2\pi n} \Big[\frac{\mathrm{f}}{\pi n}\, e^{\mathrm{f}} \cos(\pi n) - \frac{\mathrm{f}}{\pi n}\Big].$$

And finally,

$$(I) = \Big[\frac{L_y}{2\pi n} \Big((-1)^n \frac{\mathrm{f}}{\pi n}\, e^{\mathrm{f}} - \frac{\mathrm{f}}{\pi n}\Big)\Big] \Big/ \Big[1 + \Big(\frac{\mathrm{f}}{\pi n}\Big)^2\Big]. \tag{6}$$

– (II)

$$(II) = \int_0^{\frac{L_y}{2}} e^{-\frac{2y}{a}} \cos\Big(\frac{2\pi n x}{L_y} y\Big)\, dy.$$

Applying also integration by parts:

$$(II) = \left[\frac{L_y}{2\pi n}\, e^{-\frac{2y}{a}} \operatorname{sen}\Big(\frac{2\pi n}{L_y} y\Big)\right]_{y=0}^{y=\frac{L_y}{2}} + \int_0^{\frac{L_y}{2}} \frac{\mathrm{f}}{\pi n}\, e^{-\frac{2y}{a}} \operatorname{sen}\Big(\frac{2\pi n}{L_y} y\Big)\, dy.$$

Using integration by parts again:

$$(II) =$$

$$= \left[\frac{L_y}{2\pi n}\, e^{-\frac{2y}{a}} \operatorname{sen}\Big(\frac{2\pi n}{L_y} y\Big) - \frac{\mathrm{f} L_y}{2n^2\pi^2}\, e^{-\frac{2y}{a}} \cos\Big(\frac{2\pi n}{L_y} y\Big)\right]_{y=0}^{y=\frac{L_y}{2}} - \frac{\mathrm{f}^2}{(n\pi)^2} \int_0^{\frac{L_y}{2}} e^{-\frac{2y}{a}} \cos\Big(\frac{2n\pi}{L_y} y\Big)\, dy =$$

$$= \left[\frac{L_y}{2\pi n}\, e^{-\frac{2y}{a}} \operatorname{sen}\Big(\frac{2\pi n}{L_y} y\Big) - \frac{\mathrm{f} L_y}{2n^2\pi^2}\, e^{-\frac{2y}{a}} \cos\Big(\frac{2\pi L_y}{2n^2\pi^2}\, e^{-\frac{2y}{a}} \cos\Big(\frac{2\pi n}{L_y} y\Big)\right]_{y=0}^{y=\frac{L_y}{2}} \frac{\mathrm{f}^2}{(\pi n)^2}(II).$$

Then, we obtain:

$$(II) = \left[\frac{L_y}{2\pi n}\Big(-(-1)^n \frac{\mathrm{f}}{\pi n}\, e^{-\mathrm{f}} + \frac{\mathrm{f}}{\pi n}\Big)\right] \Big/ \left[1 + \Big(\frac{\mathrm{f}}{\pi n}\Big)^2\right]. \qquad (7)$$

– (III)

Lastly,

$$(III) = \int_0^{\frac{L_y}{2}} \cos\Big(\frac{2\pi n}{L_y} y\Big)\, dy = \left[\frac{L_y}{2\pi n} \operatorname{sen}\Big(\frac{2\pi n}{L_y} y\Big)\right]_{y=0}^{y=\frac{L_y}{2}} = \frac{L_y}{2\pi n} \operatorname{sen}(n\pi) = 0. \qquad (8)$$

Replacing the expressions (6), (7) and (8) in (5), we deduce:

$$a_n = \frac{1}{2\pi n}\Big((-1)^n \frac{\mathrm{f}}{\pi n}\, e^{\mathrm{f}} - (-1)^n \frac{\mathrm{f}}{\pi n}\Big) \Big/ \Big(1 + \Big(\frac{\mathrm{f}}{\pi n}\Big)^2\Big)$$

$$a_n = \frac{(-1)^n \mathrm{f}}{2((\pi n)^2 + \mathrm{f}^2)}\Big(e^{\mathrm{f}} - e^{-\mathrm{f}}\Big).$$