Sensitivity and robustness of transparent boundary conditions.

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1. Introduction.

So-called transparent boundary conditions were derived in McDonald, 2001, (M1) and tested there for the linearized shallow water equations. They were tested for the full shallow water equations in McDonald, 2002, (M2). The results were sufficiently encouraging to consider using these boundary conditions in the multi-level HIRLAM. Before doing so, however, it is advisable to ask the question: are they robust?

2. Sensitivity.

In order to see why robustness may be a problem we show in this section that the solution to the linearized shallow water equations,

$$\frac{\partial u}{\partial t} + u_0\frac{\partial u}{\partial x} + v_0\frac{\partial u}{\partial y} + \frac{\partial \Phi}{\partial x} - fv = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u_0\frac{\partial v}{\partial x} + v_0\frac{\partial v}{\partial y} + \frac{\partial \Phi}{\partial y} + fu = 0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial t} + u_0\frac{\partial \Phi}{\partial x} + v_0\frac{\partial \Phi}{\partial y} + \Phi_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{2.3}$$

is very sensitive to the errors in the choice of boundary conditions.

To do so, let us use these equations to model the advection of a balanced bell through the northwestern corner of a square area at a velocity $u_0 = -50m/s, v_0 = -50m/s$ as displayed in figure 1. For that integration the value of Φ at the northwestern corner was *imposed*: $\Phi(20,20) = \Phi^h$. (The equations were discretized on a 20×20 grid). If, instead we mistakenly *extrapolate* and use $\Phi(20,20) = 2\Phi(19,19) - \Phi(18,18)$ the resulting integration is noisy and inaccurate, as is shown in figure 2. It must be emphasized that no other changes were made aside from the seemingly innocuous mistaken assignment of the corner value of Φ.

Obviously, this integration can be very sensitive to boundary errors. This prompts the question: is this an indication that lack of robustness will be a problem?

3. Robustness.

Steep orographic features on the boundary sometimes cause problems in limited area integrations. In this section we address the question: will the orography spoil everything for the transparent boundaries?

If we include the orography the full shallow water equations become

$$\frac{du}{dt} + \frac{1}{acos\theta}\frac{\partial \Phi}{\partial \lambda} - \left(f + \frac{utan\theta}{a}\right)v = 0, \tag{3.1}$$

$$\frac{dv}{dt} + \frac{1}{a}\frac{\partial \Phi}{\partial \theta} + \left(f + \frac{utan\theta}{a}\right)u = 0, \tag{3.2}$$

$$\frac{d(\Phi - \Phi_s)}{dt} + \frac{(\Phi - \Phi_s)}{acos\theta}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial vcos\theta}{\partial \theta}\right) = 0 \tag{3.3}$$

If we repeat the nested integrations of M2, but now with steep orography on the boundary of the guest area will they still be stable and accurate, and will the transparent boundary still give better forecasts than the traditional Davies (1976) relaxation scheme? To answer this question the tests described in M2 were repeated with the nesting set-up exactly as described in its section 3. The host model coarse fields were interpolated to the guest model fine grid using the bi-cubic spline to interpolate in space and the cubic spline to interpolate in time. As can be seen from figure 3 the guest area orography should provide a reasonably severe test of the robustness of the boundary strategies in the presence of steep orography.

The results of the tests were that the forecasts were stable and accurate. Thus, for this data set, at least, we can conclude that lack of robustness is not an issue.

In order to get some insight into what is happening let us look at the errors over the whole of the guest area. Looking at the coarse mesh forecast we can see that its errors are concentrated in the north and west of the guest area; see figure 4 which shows the difference between the coarse and fine mesh 12h forecasts of the host model displayed on the guest area. The difference between the guest forecast using 'balanced b.c.' and the fine mesh host forecast is shown in figure 5. First notice there is no extra 'noise' visible in the vicinity of steep orography. Second, the errors have been reduced significantly where they are large. Compare that with the same chart for the Davies (1976) 'relaxation b.c.', figure 6, whose errors are significantly larger.

4. Discussion.

What about internal gravity waves for which the advection velocity can be greater than $\sqrt{\Phi}$? Now, we must impose 3 fields at inflow and none at outflow in order to have well-posed b.c. for the shallow water equations. I have derived a set of 'transparent b.c.' and programmed and tested them for the linearized shallow water equations. The integrations are stable and accurate.

Will the physics spoil everything? We will not know the answer to this question until we program up the multi-level equations. However, we can partially address the following question in a shallow water context. Will horizontal diffusion cause insuperable problems? With its inclusion in the shallow water equations we must impose 3 fields at inflow and 2 at outflow, and worry about the formation of 'numerical boundary layers'. This is the final item on the shallow-water agenda. Assuming there are no problems the programming and testing of the multilevel model will follow.

A final (unrelated) speculation: in the reference system we should think about treating passive scalars as follows: impose at inflow, extrapolate at outflow. The latter is easy using S.L. scheme because it uses upstream information.

5. References.

Davies, H.C., 1976: A lateral boundary formulation for multi-level prediction models. *Quart. J. Roy. Meteor. Soc.,* **102**, 405-418.

McDonald, A., 2001: Well-posed boundary conditions for semi-Lagrangian schemes: the two-dimensional case. *HIRLAM technical report* **47** .

McDonald, A., 2002: Testing transparent boundary conditions for the shallow water equations in a nested environment. *HIRLAM technical report* **54** .

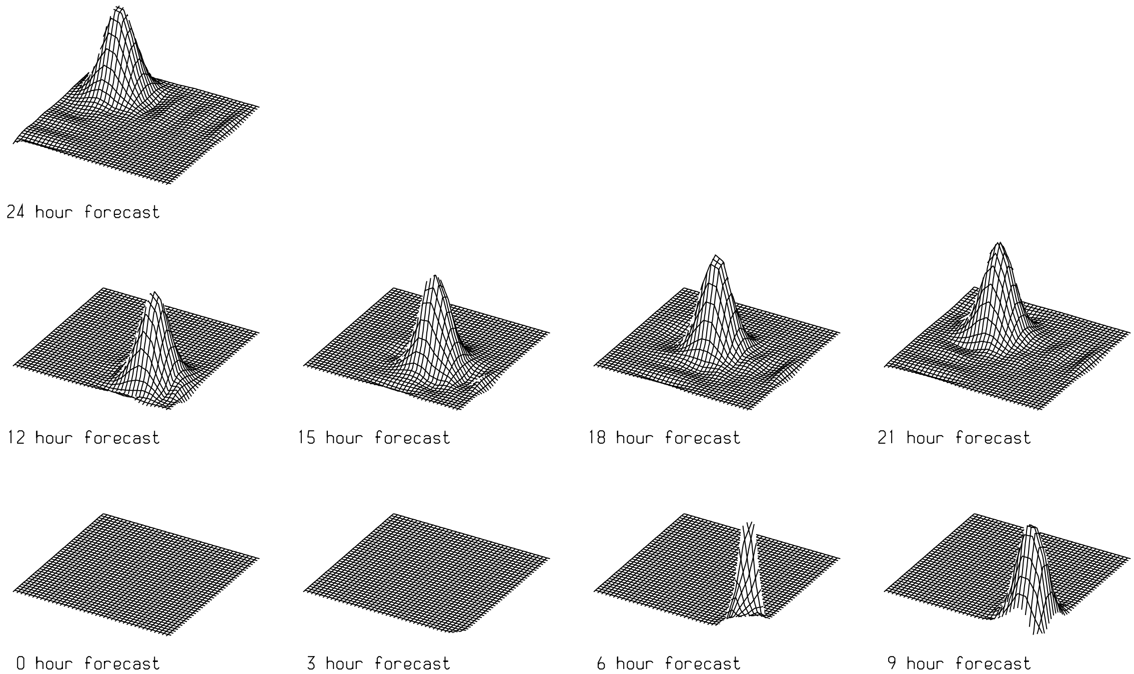

Figure 1: *Advection of a bell shape into the area* imposing Φ *at the entry point corner.*

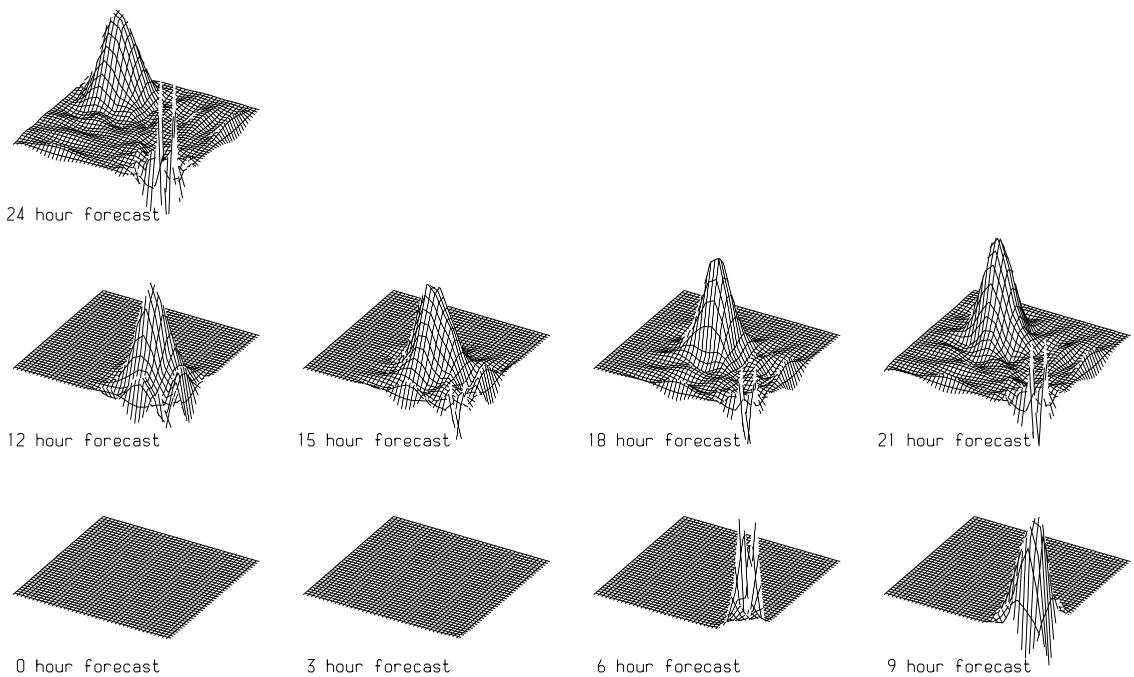

Figure 2: *Same as FIG. 1 but now* extrapolating Φ *from the interior to the entry point corner.*

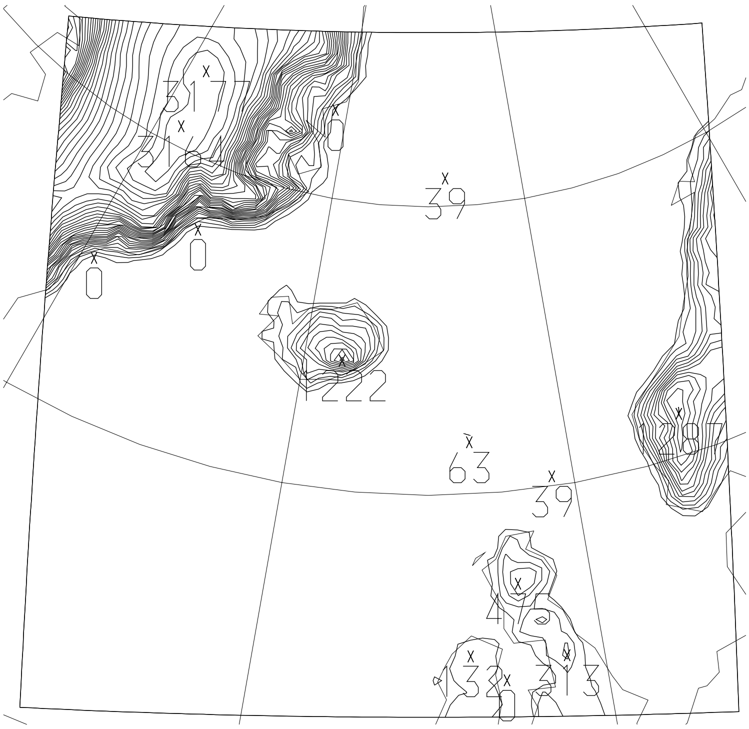

Figure 3: *The guest area and its orography.*

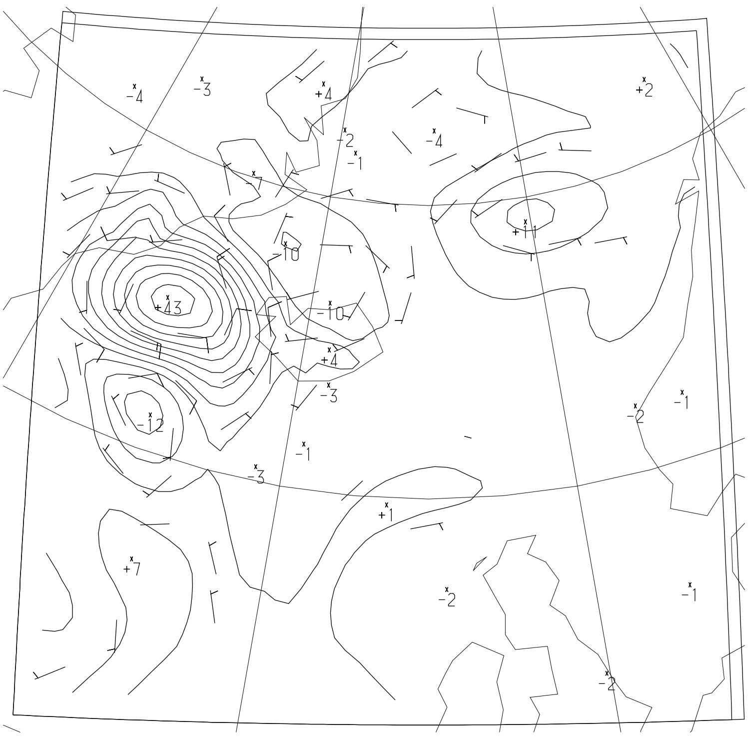

Figure 4: *The difference between the coarse and fine mesh 12h height forecasts of the host model displayed on the guest area.*

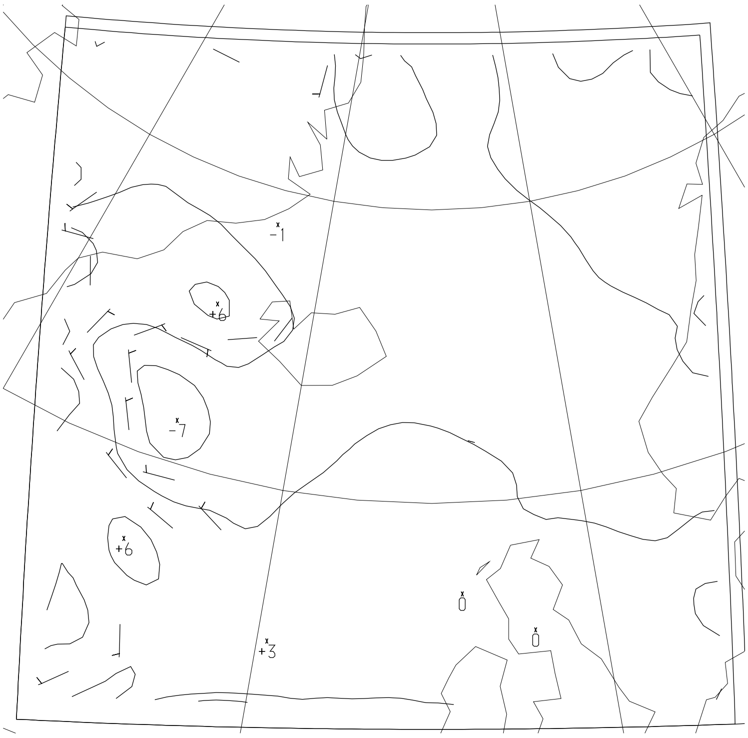

Figure 5: *The difference between the guest forecast using 'balanced b.c.' and the fine mesh host forecast.*

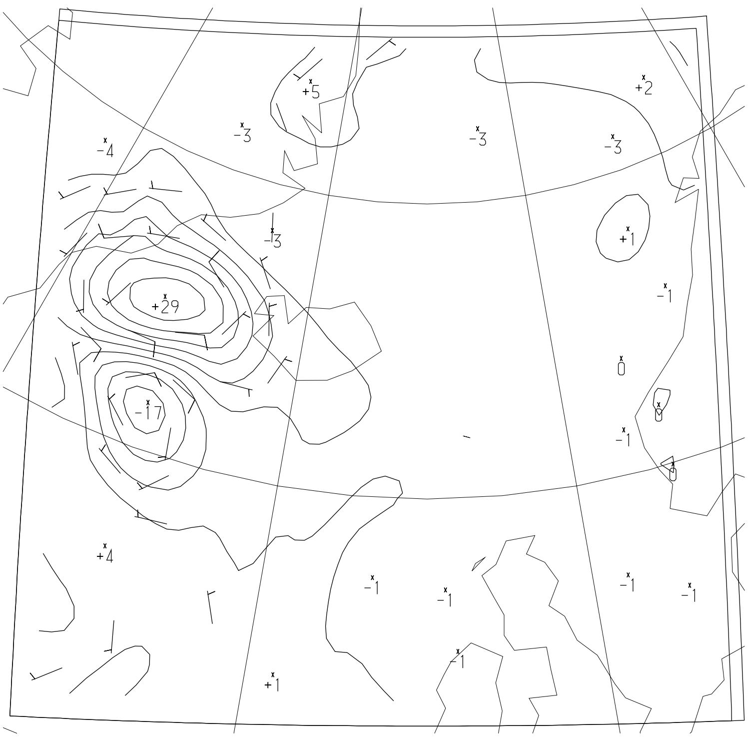

Figure 6: *The difference between the guest forecast using 'relaxation b.c.' and the fine mesh host forecast.*