LATERAL BOUNDARY CONDITIONS; A PROGRESS REPORT.

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1. Introduction.

Consider the 'D' area of the Danish Meteorological Institute, as described on page 32 of HIRLAM newsletter 35. It is approximately 900km across. It is used for running 36h fine mesh forecasts. In 36h a rapidly moving (at 25km/h) deepening depression would cross from one boundary to the other. This means that from hour 18 onwards the forecast over Denmark, the area of interest, would be dominated by the information crossing the inflow boundary rather than the initial conditions. To state it controversially: for small area fine mesh forecasts the boundaries are more important than the analysis in one of the most important meteorological scenarios (because of its potential for devastation). Therefore, it is important that we make our boundary treatment as accurate as possible.

To accomplish this there are two minimum requirements: we would like the boundaries of our limited area model to be *well-posed* and *transparent*. By 'well-posed' we mean, basically, that our boundary treatment does not amplify errors. By 'transparent' we mean that, first, all waves approaching the boundaries from the interior of the limited area should exit without reflection; and second, in a nested environment, where the boundary data are being supplied by a 'host' model, all meteorologically important waves impinging on the boundaries from the exterior of the limited area should enter without their amplitude or phase being changed and without exciting spurious high frequency waves (noise).

Full transparency is probably unattainable in most practical integrations, but there exist a variety of methods for making the boundaries 'as transparent as possible'. The basic idea is as follows. First we analyse a linearized version of the equations of motion. From this analysis we separate the waves leaving the area from those entering the area. We then use this information to force the boundary fields to obey equations which describe only *uni-directional* waves travelling *out* of (or into) the area. For example, when using the primitive equations, we would like to impose boundary conditions which let all the gravity waves and meteorological waves out of the area, while simultaneously allowing the meteorological waves into the area without letting in any gravity waves. In principle, this is an attainable objective, although there may remain a low level of wave reflection, as was stated above.

McDonald (2001b) explains how to derive transparent boundary conditions of various levels of accuracy for the shallow water equations. The accuracy depends on two parameters. The first is the angle which the group velocity of the waves makes with the boundary. Full transparency is attained when the wave is perpendicular to the boundary. The wider the angle the more terms needed to maintain transparency. The second parameter is $f/(K\sqrt{\Phi_0})$ where f is the Coriolis parameter, Φ_0 the mean geopotential height and K the wave number. We can expect $f/(K\sqrt{\Phi_0}) \ll 1$ for most waves in a limited area model, and in particular for the fast-moving short wavelength modes which we especially do not want to be reflected from the boundaries. Two of these transparent boundary conditions will be tested in section 2.

2. Numerical testing.

Detailed testing of different well-posed boundary conditions was carried out in McDonald (2001a) and McDonald (2001b). A subset of these tests is presented here to partially motivate the discussion in section 3.

Robert and Yakimiw (1986) point out that, since most boundary strategies contain difficulties that cannot be easily identified when they are considered in the framework of realistic models, it is a good idea to first test them on simple problems with known solutions. (If our boundary treatment will not work for a simplified sub-system it certainly will not work for the full non-linear system). That is the philosophy we adopt. Therefore, we will take the *linearized* shallow water equations and compare the forecasts using our discretization and boundary treatment with the known solution for an *advection* case whose analytical solution is known at all times.

We will test four different sets of boundary conditions. Three are well-posed: (1) Φ imposed on all boundaries and the tangential velocity, v_T, imposed at inflow boundary points. (2) $\Phi - \sqrt{\Phi_0}v_N$ imposed at all boundary points and v_T imposed at inflow boundary points; (v_N is the normal velocity at the boundary). (3) $\Phi - \sqrt{\Phi_0}v_N$ imposed at inflow and $d(\Phi - \sqrt{\Phi_0}v_N)/dt = 0$ imposed at outflow; v_T imposed at inflow boundary points. (4) Over-specify the boundaries and damp the resulting errors by relaxing the fields in the neighbourhood of the boundary toward the host model fields using the Davies (1976) relaxation scheme. This corresponds to our present HIRLAM boundary treatment.

For the following demonstrations $\Delta x = \Delta y = 100$km $L_x = L_y = 10000$km (there are 101 grid points in each direction). Below we model features of size $L_x/10$. Thus we we will generate waves with a typical wavelength of 1000km or less. $\Phi_0 = 5000g$, $\breve{\Phi} = 500g$ and $f = 0.729 \times 10^{-4}s^{-1}$. We use $\Delta t = 15$min.

The following is an analytical solution of the linearized shallow water equations which describes the advection of a bell shape with a constant velocity (u_0, v_0) starting from a position (x_s, y_s). Thus we have an exact answer with which we can compare our integrations. Also, the system is in geostrophic balance and the analytical divergence is always zero providing us with an additional useful test of the efficacy of our discretization.

$$\Phi(x,y,t) = \Phi_0 + \breve{\Phi}\ exp\left[-\left\{\frac{(x - x_s - u_0t)}{(L_x/10)}\right\}^2\right]\ exp\left[-\left\{\frac{(y - y_s - v_0t)}{(L_y/10)}\right\}^2\right] ;$$

$$v(x,y,t) = \frac{-2(x - x_s - u_0t)}{f(L_x/10)^2}\Phi(x,y,t) \ ; \quad u(x,y,t) = \frac{2(y - y_s - v_0t)}{f(L_y/10)^2}\Phi(x,y,t).$$

With our choices of the various parameters the bell can be thought of as a geostrophically balanced sharp pseudo-meteorological feature. (The maximum geostrophic wind is 57.5m/sec.).

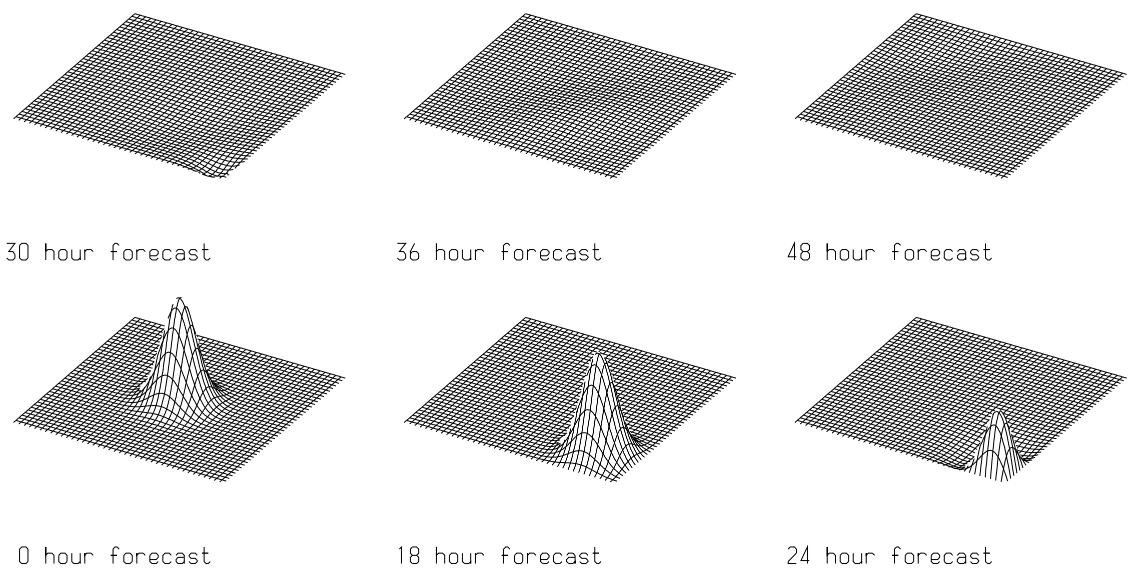

Figure 1: *The advection of a bell shape out of the area using option (1): imposing* Φ *on the boundaries.*

We address the question: when the bell exits the area how big are the reflections from the boundary? In particular, if the imposed boundary conditions are wrong how badly behaved is the system? (This models an unavoidable operational situation in which the host model fields are inaccurate at outflow). In order to make this a truly two-dimensional test let us start with the bell at the centre of the area and advect it so that it exits through a corner; thus $(x_s, y_s) = (L_x/2, L_y/2)$ and $(u_0, v_0) = (50, 50)ms^{-1}$, so that it leaves the area through the corner defined by $x = L_x, y = L_y$.

TEST(1): Φ imposed. We impose $\Phi(0,y,t) = \Phi(L_x,y,t) = \Phi(x,0,t) = \Phi(x,L_y,t) = 5000g$. We impose $v_T = 0$ on the inflow boundaries at all times. We extrapolate v_T at outflow boundaries. The initial state and the 18h, 24h, 30h, 36h, and 48h forecasts are displayed figure 1. As can be seen, the bell disappears almost without trace. There is no sign of instability nor of two-grid noise. Looking carefully at the 36h and 48h forecasts we can just see a feature with a dominant wavelength of about $L_x/2$. This feature is not in geostrophic balance as can be seen from 'x's in the the graph of mean absolute divergence shown in figure 3. The rms difference between the 48h forecast and the analytical solution is shown in column 2 of table 1.

	TEST(1)	TEST(2)	TEST(3)	TEST(4)
rms for Φ	6.75	1.36	0.15	0.89
rms for u	0.227	0.058	0.003	0.044
rms for v	0.247	0.064	0.004	0.032

TABLE 1. Root mean square difference between the 48 hour forecast and the analytical solution; bell exiting .

TEST(2): $\Phi - \sqrt{\Phi_0}v_N$ imposed. We impose $\Phi - \sqrt{\Phi_0}v_N = 5000$g on all boundaries and $v_T = 0$ at the inflow boundaries. When there is outflow, v_T is computed by extrapolation.

We repeat TEST(1) with these new boundary conditions. The forecasts are shown in figure 2. Looking at these charts we conclude that there is no instability and almost no reflection at the boundary. In fact, there is a small amount of error. The rms difference between the 48 hour forecast and the analytical solution is shown in column 3 of table 1. The errors have been reduced significantly, but there remains some reflection. The mean absolute divergence shown as circles in figure 3, still shows a big increase as the bell reaches the boundary but subsequent to that the geostrophic balance is much better than for TEST(1).

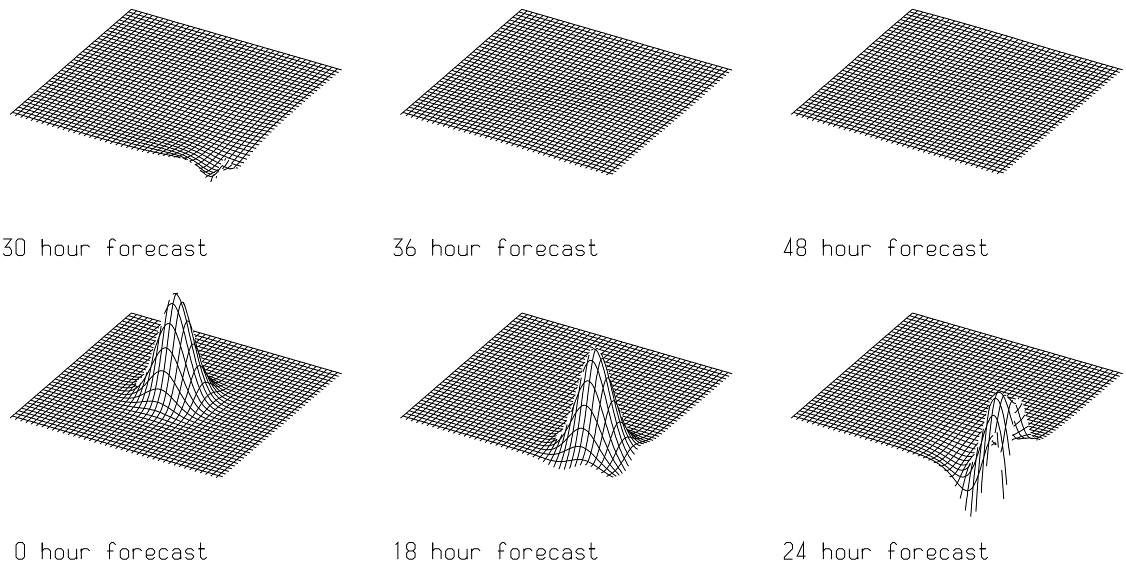

Figure 2: *The advection of a bell shape out of the area using option (2): imposing $\Phi - \sqrt{\Phi_0}v_N$ on the boundaries.*

TEST(3): $\Phi - \sqrt{\Phi_0}v_N$ imposed at inflow and $d(\Phi - \sqrt{\Phi_0}v_N)/dt = 0$ imposed at outflow. We now impose $\Phi - \sqrt{\Phi_0}v_N = 5000$g at inflow and $d(\Phi - \sqrt{\Phi_0}v_N)/dt = 0$ at outflow, treating v_T in the same manner as always. Is the boundary is now more transparent ? We repeat TEST(2) with this new condition at outflow. The forecasts (not displayed) show greater transparency. This is demonstrated by the reduction in rms error (see column 4 of table 1) and good geostrophic balance throughout the forecast; see the dots in the graph of mean absolute divergence, figure 3.

TEST(4): Over-specified boundaries. Have we beaten the competition? We repeat our experiments but now over-specifying the boundary fields and relaxing them, as described in section 3.3 of McDonald (2001a), toward the following host model values: $\Phi = 5000g, u = 0$ and $v = 0$. From column 5 of table 1 we see that this scheme is very effective in damping the reflected waves. However, from figure 3 we see that there is a large increase in mean absolute divergence as the bell goes through the boundary.

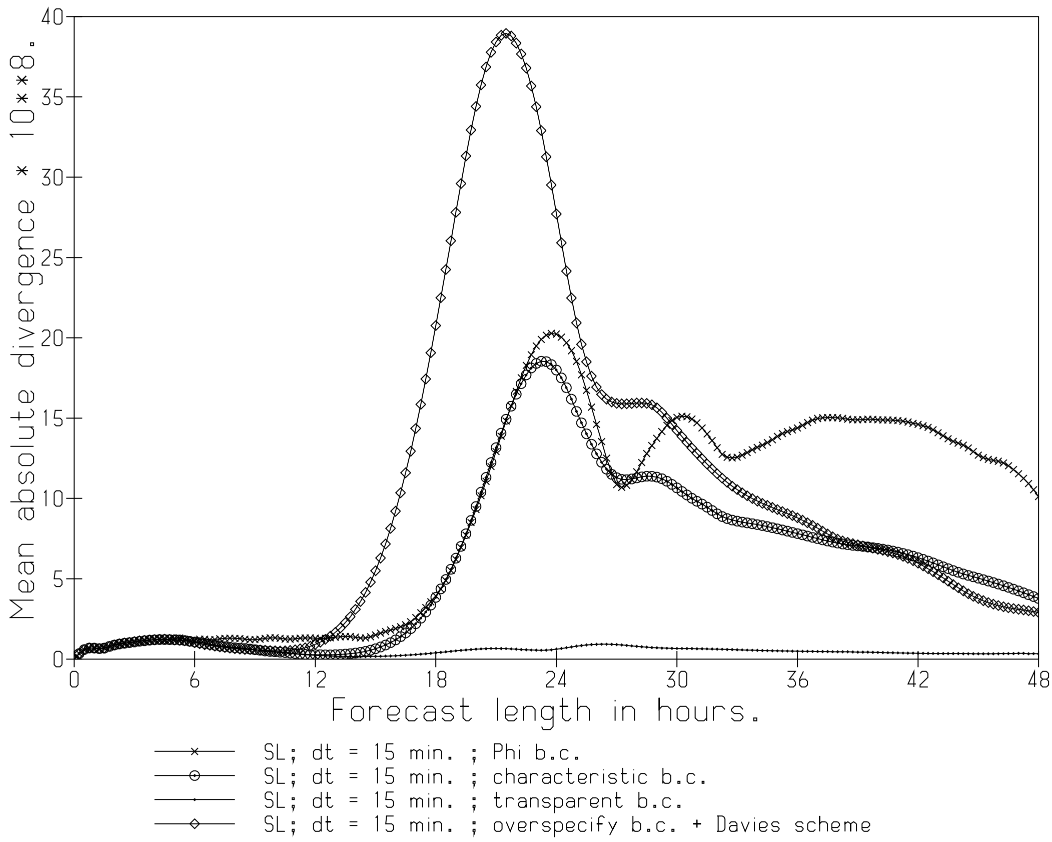

Figure 3: *Graph of the mean absolute divergence multiplied by* 10^8. *'x's :* Φ *imposed; circles:* $\Phi - \sqrt{\Phi_0} v_N$ *imposed; and dots:* $\Phi - \sqrt{\Phi_0} v_N$ *imposed at inflow and* $d(\Phi - \sqrt{\Phi_0} v_N)/dt = 0$ *imposed at outflow; diamonds: over-specify and use boundary relaxation.*

3. <u>Discussion.</u>

In this section the phrase 'experimentally well-posed' will be used to produced forecasts which had no visible two-grid noise on the 10-day chart and for which the mean absolute divergence was not increasing significantly with time at day 10. Based on the tests described in McDonald, 2001a and 2001b we can draw the following conclusions.

(1). It is possible to invent experimentally well-posed semi-Lagrangian discretizations of the linearized shallow water equations in two dimensions. Trajectory truncation, not surprisingly, is inaccurate when the departure point is far outside the boundary. Time interpolation works well despite the many extrapolations involved in its implementation. It is fair to say that it was the most accurate scheme tested.

(2). We have also seen that experimentally well-posed Eulerian leapfrog discretizations are feasible. The is some residual lack of balance which remains a puzzle.

(3). Well-posed boundaries of themselves, although attractive from a theoretical point of view may not be particularly useful. They guarantee a stable solutions but, as we have seen, they do not guarantee accurate solutions.

(4). The three sets of boundary conditions which we tested were experimentally well-posed. They showed varying degrees of transparency in agreement with our analysis. When we imposed Φ on all boundaries we saw that adjustment waves were reflected at the boundary in our adjustment experiments, and that 'trailing waves' developed in our advection experiments. When, instead, we imposed $\Phi - \sqrt{\Phi_0}v_N$ on all boundaries we obtained not only stable forecasts, but also their accuracy was improved dramatically. The adjustment wave reflection was reduced significantly, and the 'trailing waves' phenomenon was eliminated. When we imposed $\Phi - \sqrt{\Phi_0}v_N$ at inflow boundaries and $d/(\Phi - \sqrt{\Phi_0}v_N)/dt = 0$ at outflow we saw that transparency improved significantly for the the advection solution at outflow.

(5) The Davies (1976) relaxation scheme is very competitive for these simple tests. It worked well at reducing gravity wave reflections and the 'trailing waves' phenomenon. It consistently beat the 'Φ-imposed' scheme as far as the rms errors were concerned, losing only to the transparent schemes, and then sometimes only marginally.

(6) We have demonstrated the technical feasibility of transparent boundary conditions for the linearized shallow water equations. An obvious next question to be addressed is: will these results hold up when the non-linear terms, with their potential for explosive growth, are included? To test this I am in the process of setting up a nested system to integrate the full (non-linear) shallow water equations using real data. This set-up will enable us to examine the accuracy and robustness of the transparent boundary conditions in a more demanding and realistic environment.

(7) Assuming these tests yield positive results, can we apply these ideas directly to a multi-level model? Oliger and Sundström (1978) established necessary conditions for the well-posedness of the linearized hydrostatic equations by doing a normal mode decomposition. For each vertical mode this projects out a shallow water equation for which the well-posed boundaries are well-known. (There is no guarantee that these will be *sufficient* for well-posedness). This gives us an idea of how to proceed for hydrostatic models. Each mode has an associated $\sqrt{\Phi_0}$, which enables us to apply the transparent boundary conditions to each of the projected shallow water equations. Once the boundaries have been updated we transform back to physical space. (For many semi-implicit models this concept is familiar. We solve the Helmholtz equation for the transformed fields).

References.

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