Technical Report

No. 41, May, 1999

Some Aspects of Non-hydrostatic Models in the HIRLAM Perspective

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May 3, 1999

1 Introduction

The purpose of this report is to provide some background for the decisions that will be taken within the HIRLAM group with respect to a future nonhydrostatic HIRLAM model. The initiative to write it came from the HIRLAM management group.

There are many ways to approach the construction of a nonhydrostatic model, and many more opinions on how one ought to do it. There is also an extensive literature on theoretical aspects, physical as well as mathematical. However, analyses of the numerical methods suggested for nonhydrostatic models are often based on simplifications and therefore only provide partial answers. We will be rather selective in what we discuss in this report, but hopefully many more opinions and viewpoints will be presented in the discussions on nonhydrostatic models that are supposed to take place in the coming months.

The opinions and viewpoints in this report are the author's, and do not necessarily reflect those of DNMI or the HIRLAM group. This report is supposed to provide a basis of a discussion within HIRLAM, hopefully with some scientific background. I have no intention of telling anyone "the truth" about nonhydrostatic models because, fortunately for the scientific community of numerical weather prediction, we need quite a bit more insight in how such models should be constructed.

HIRLAM is today used operationally with horizontal grid sizes ranging from approximately 50 km down to 10km. Since the global models now use a grid size of approximately 50 km, there is an obvious requirement for operational limited area models to work well for grid sizes down to a few km. What the appropriate grid size will turn out to be is dependent on many things, for example the area, and the physics, and cannot be decided until more practical experience has been gained.

Almost everyone agrees that with a grid size of a few kilometers, the hydrostatic assumption is not very accurate any longer. Hence in this report I will not discuss whether we need a nonhydrostatic model or not. This does not imply that a nonhydrostatic model will be the solution to every numerical weather forecasting problem, only that such a model will give us a set of more versatile tools.

Only a few of the current nonhydrostatic models are used for operational purposes today. This is not an indication that these models are inferior to the hydrostatic ones, the reason is more historic: we have now reached a level of competence in the numerics of such models, and have sufficient computer power that it is practical to use them. On the other hand, nonhydrostatic models are in extensive use for research, for example in the study of orographically forced wave phenomena.

The ALADIN model in France is used operationally at approximately 10 km resolution, and is also used in the Eastern Europe in the LACE project. The German Lokal-Modell (LM) is now in pre-operational testing at 7.5 km, and the UM in the UK with new dynamics is also in the same state. Of the models not being used operationally, the Canadian MC2 has shown very good results, see e.g. [2] for an example of nesting from 50 km down to 0.4 km, and [3] for a 10km run over North America. So one can say that the experience in the operational use of nonhydrostatic models is increasing. The verification results are however not always better than a hydrostatic competitor. One other characteristic of nonhydrostatic models is that the numerical methods are developed rapidly compared to the hydrostatic models where many details are more or less settled. Of course these two aspects may be related. So things are going to change numerics-wise in almost all the models, and that means one has to look carefully trying to find the 'right' schemes, and be flexible with respect to changing parts of the numerical formulation.

One comment on terminology used for high resolution models: the terms mesoscale and microscale, with subdivisions indicated by α, β and γ are often used. There are different opinions on what the resolution range for this scale classes are, so I have avoided using them altogether and instead indicate the resolution in terms of kilometers.

2 Existing NH models and their use

In this section we will describe briefly some of the nonhydrostatic models. There are many more around, but we tried to pick some of the models that could be relevant for HIRLAM. All the models discussed below are complete models in the sense that they come with fairly new physics packages, and can be run together with modern analysis schemes.

The discussion will be based on characteristics of the models such as discretization details, but also on the author's and other people's opinion on the quality of the results, numerical approach etc. These aspects may be controversial, but again the purpose of this report is to provide a basis for discussion.

The governing equations for the dynamics of each model will not be discussed in detail because for most models they are quite similar: momentum equations for all wind components, a pressure equation (or continuity equation), an energy equation and transport equations for moisture an cloud water. The semi-implicit solution of this set of equations boils down to solving a Helmholtz equation, and that is an important point we will return to. Note that from a mathematical point of view, the set of equations used in nonhydrostatic models is easier to work with than the hydrostatic counterpart because well-posedness with many classes of boundary conditions can be proved in a fairly straightforward way using the theory in [20]. However, the regularity of the continuous solution is still an open question. As is well known the hydrostatic equations are problematic with respect to well-posedness, and only very recently have these theoretical issues got some attention. The well-posedness issues may seem fairly remote from numerical weather prediction, but it is important to know that the system of equations we are approximating does have a solution, and that we can speak about the convergence of the numerical solution in some way.

This report will not deal with physics packages used in the different models in any detail, because this will require a report of its own and because the author does not feel competent to come up with something meaningful as a basis for discussion. However, the impression is that different models are using the same physics packages as other (hydrostatic) models, which may indicate that the real discussion about physics at finer scales has not really started.

The same argument goes for the data assimilation and analysis schemes: We will not discuss it, and about the only thing that will be said here is that this is a challenge that has to be addressed seriously in order to produce high quality results for finer scale models.

2.1 The MM5 model

The Fifth Generation NCAR/Penn State Mesoscale Model (hereafter MM5) originates from the model development in the 1970's by Anthes at Penn State. The model has evolved into the fifth generation of which version 1 was released in 1994, and the current version 2 came late 1997. There have been many releases of version 2, the latest (no.12) came at the beginning of April, 1999. Since the model runs on many platforms, from simple PCs to massively parallel computers, a special parallel version of the model has been developed. MM5 is the most widely used non-hydrostatic model today, and there is a large and active user group taking part in the development. The model is mainly used for special studies with high resolution. A thorough description of the model is given in [7], but there exists also extensive documentation on-line and course notes from MM5 courses regularly held at NCAR.

The governing equations for the model are the compressible Euler equations with a pressure-based σ-coordinate:

$$\sigma = \frac{p - p_t}{p_s - p_t} \tag{2.1}$$

as the vertical coordinate, where p_s and p_t are the surface and top pressure respectively.

The horizontal discretization is on a Arakawa B-grid and the usual finite differences in the vertical.

The time-stepping is done by a split-explicit scheme, but an implicit scheme is used for vertical diffusion and vertical propagation of sound waves.

The MM5 model can be nested both 2-ways and 1-way at a ratio of 1 : 3 at several levels. The nesting is quite flexible, one can for example use overlapping subgrids at the same level. There is a Davies type boundary relaxation at the lateral boundaries and a rigid lid at the top.

The physics package in MM5 is extensive and contains many options for ground parameterization, radiation, convection, condensation and turbulent boundary layer. Several tests have been performed to check how different physics options perform at different scales.

The parallel performance of MM5 is quite good, but it seems that the scalability is not so good as MC2. (This issue is now being addressed, by the same person that did the parallelization work for MC2). The two models are comparable in performance for relatively few processors.

MM5 version 2 now also contains functions to read analyses from different sources as input, e.g. GRIB-formatted analyses from ECMWF and NCEP, which implies that the model is more flexible in use.

The MM5 model is not in operational use, so it is difficult to say what the typical resolution can be if one should use it operationally. The German LM will be used at resolution 7.5km and tests have been performed with resolution 2.7km. It is not unreasonable to say that MM5 could be used in much the same way. The experiments with MM5 have been conducted with resolutions ranging from 50km down to 0.5km.

2.2 The Lokal-Modell

The Lokal-Modell (LM) is the next generation LAM of DWD. The prototype of the model was developed with reference to the formulation of MM5, but some new features have been implemented. The governing equations are the compressible Euler equations, formulated with thermodynamic variables as perturbations from a base state. The model can use either pressure-based η-coordinates

$$p_0 = A(\eta) + B(\eta)p_0^s, \tag{2.2}$$

where p_0 is the base state pressure and p_0^s is the reference pressure at the surface, or height-based Gal-Chen coordinates, where the heights are computed by inverting the transformation

$$z = a(\mu) + b(\mu)h. \tag{2.3}$$

The LM uses rotated spherical coordinates with a Arakawa C-grid in the horizontal and a Lorenz staggering in the vertical.

The original advection scheme was Eulerian split-explicit, but recently a fully 3D semi-implicit scheme has also been implemented, see [6].

There is a Davies-type absorbing layer at the lateral boundaries and rigid lid at the top. A version with radiation conditions described in [28] is under development.

The LM was initially using the Deutschland-Modell (DM) physics package, but now new LM physics is in preoperational testing with the following features: prognostic turbulent kinetic energy, explicit ice phase, biosphere model, surface model based on the agrometeorological model AMBETI.

The planned configuration is to use the global model GME at 30 km and LM at 2.5km with 50 vertical layers for both models. The LM is now in preoperational testing phase with a $325 \times 325 \times 35$ grid with horizontal resolution 7.5km. It is possible to nest LM with itself with 2-way nesting in the same way as MM5.

The model is programmed in Fortran 90 and parallelized with MPI. The performance of the model scales well with the number of processors of the T3E.

Comparisons with other models in terms of quality of results and run time have been performed in a study together with the MRI-NHM, a nonhydrostatic model developed at the Meteorological Research Institute in Japan, but only on mountain wave runs with the Smith mountain on a C-98. The results are reported in [9], and the LM performs better than MRI-NHM down to a grid size of about 2km, somewhat dependent on boundary conditions. A recent experiment to compare the split-explicit advection with the semi-implicit version has been performed on a $200 \times 150 \times 35$ grid with approximately 2.7 km gridsize. The area chosen was in Southern Germany with some steep topography so orographic forcings were to be expected. Real analyzed data were used to initialize the model, and the run was performed with full physics. The results showed only small differences in the computed fields and very similar performance. The time per forecast hour on a 128 node T3E varied from 120s to 150s.

The performance of the preoperational testing configuration is as follows: A 24h forecast is completed in 1894 s. on a 200 processor T3E, which gives approximately 79s per forecast hour and an estimated 11.4 Gflops/s sustained performance.

2.3 The ALADIN model

ALADIN, or more precisely ARPEGE/ALADIN is the limited area version of the ARPEGE/IFS system developed by Méteo-France and partners. The model is operational in France and in the countries within the LACE project.

The numerical schemes for the dynamics of the model are described in [13]. ALADIN uses a hybrid η coordinate in the vertical, and the formulation of the governing equations follows the work of Laprise [14]. The advantage of this approach is that the governing equations are formulated in an η system based on the hydrostatic pressure, hence the formulation contains both a nonhydrostatic model and a hydrostatic one (obtained by just omitting the vertical momentum equation). This switching between models can of course be implemented just by changing an input parameter. The Laprise approach will also be discussed below and details will be presented in Appendix A. ALADIN differs from the original Laprise formulation in only one detail, ALADIN uses a scaled pressure difference

$$\mathcal{P} = \frac{p - \pi}{\pi^*}$$

instead of the pressure itself as a variable to avoid large cancellation errors in the vertical momentum equation. In the formula above, π represent the hydrostatic pressure, and π^* a reference hydrostatic pressure.

The discretization is spectral in the horizontal and finite difference in the vertical. Fourier spectral collocation in the horizontal is used for all prognostic variables on an unstaggered grid, while the Lorenz staggering is used in the vertical. The time discretization is semi-implicit, and since the Helmholtz operator is diagonal in Fourier space, the cost of the solution of the Helmholtz equation amounts to the cost of the transforms. On the other hand, it has turned out that because of instabilities the full velocity divergence has to be taken implicitly, resulting in a full 3D Helmholtz nonlinear Helmholtz equation. This is solved by applying an outer fixed-point iteration.

Examples of the performance and quality of the model are presented in [13]. Since the model is operational, verification scores can be inspected to check the models quality. The model now runs with a $288 \times 288 \times 31$ grid at 9.92km horizontal resolution and timestep of 170s. A 24h forecast is completed in approximately 10 minutes on 4 processors of a Fujitsu VPP-700E. This corresponds to a sustained gigaflop rate of approximately 5.0.

2.4 The MC2 model

The Canadian MC2 model has been developed over the last 8-10 years by Recherche en Prevision Numerique (RPN) and Universite a Quebec a Montreal (UQAM), and the 'father' of this model was Andre Robert. the model is described in [2], but the details in the numerical schemes are discussed in [5].

The model uses the compressible Euler equations formulated with the Gal-Chen vertical coordinate:

$$\zeta(X, Y, z) = \frac{z - h(X, Y)}{H - h(X, Y)} H, \tag{2.4}$$

where H is the model top and $h(X, Y)$ is the height of the topography. A generalized vertical coordinate is obtained by applying a mapping $Z(\zeta)$ to ζ, where $Z(\zeta)$ is a strictly monotonic function in the range $[0, H]$.

The space discretization is on an Arakawa C-grid in the horizontal and uses Lorenz staggering in the vertical. MC2 is using a semi-implicit semi-Lagrangian scheme for advection, see [5] for the details. The efficiency of semi-Lagrangian schemes on larger and medium scales is well-proven, but realistic comparisons between Eulerian and semi-Lagrangian methods in terms of efficiency on smaller scales have not been performed. However, preliminary comparisons with MM5 and LM indicates that in terms of performance indicate that MC2 is fully competitive.

The latest version of the adiabatic kernel, see [4] for details, contains improvements in the numerical schemes, for example transparent lateral boundary conditions, full 3D Helmholtz equation solved with nonsymmetric linear equation solver, and efficient trajectory calculation. We will return to the 3D Helmholtz equation in the next section because it seems to be important for all models to be used on finer scales.

MC2 is using the comprehensive RPN physics package also used by the other Canadian models, see [2] for a brief description, and a special microphysics package, [21].

The MC2 model includes nesting capabilities, see [2] for details, and this is demonstrated in the same reference for a case with 4 levels of nested models, resolution ranging from 50km to 0.4 km. There are also results from measurement campaigns and programs, for example the Mesoscale Alpine Program (MAP). During the MAP Special Observation Period in 1999, the MC2 model will be used at resolution 2km on a $500 \times 400 \times 41$ grid covering most of the Alpine massif. The model also works well when running theoretical cases, such as flows over (artificial) mountains. The model is parallelized using MPI. The high-performance aspect of the model is shown in a 10km forecast over North America, see [3].

2.5 The Unified Model

The Unified Model(UM) is the model system developed by UK Met.Office and consists of a global model and a limited area model. The so-called New Dynamics for the UM has been developed over the last two years, is a nonhydrostatic system for both the global and the limited area model. The governing equations are the compressible Euler equations, and the model uses a Gal-Chen height-based vertical coordinate, much like MC2 and LM.

The spatial discretization is on a Arakawa C-grid with Lorenz staggering in the vertical. The advection is done by the semi-Lagrangian method described by Bates [1]. The formulation of the discrete equations is done in perturbation of the prognostic variables, i.e. the difference between the variable at the current and previous time. The semi-implicit scheme is performed by solving a fully 3D Helmholtz equation incorporating all components of the pressure gradients.

The physics is much like the Hadam4 climate physics from the Hadley centre, but specific developments aimed at the limited area version, for example new PMSL calculations, are being carried out.

The UM is fully parallelized and is using the communication-on-demand technique of Skålin [22], and the performance results for a $432 \times 236 \times 38$ grid are very good. To give an indication of the performance, climate runs are performed with 3.6 years in 1 day on a 72 processor Cray T3E.

The code is still under development, but the formulation and the basic code is considered to be stable and almost complete. Testing is going on, and problems with , for example, lateral boundary conditions and vertical winds are being addressed and solved.

3 Models under development

There are quite a few NWP models under development in many countries, and it is impossible to review even a small selection of these. Most of the models do not aim at being an operational NWP model, but are rather used for research purposes. In this section we will describe briefly one model under development, namely the Tartu model. Because it has such strong connections with the HIRLAM group, the development is supported (at least encouraged) by the HIRLAM management. Moreover, it is using the HIRLAM code as a building block.

3.1 The Tartu model

The Tartu model, or rather the Tartu project, is a model developed at the Tartu Observatory. The development is headed by Rein Rõõm, and the project was started in 1993. The development group has close connections with SMHI and FMI. The HIRLAM code is used as the starting point for the code development. A comprehensive description of the background for the model is given in [16], its current status is reviewed in [17].

The governing equations are the usual Euler compressible equations formulated in a true pressure-based vertical coordinate, so it is different from the Laprise approach. The formulation is based on the works in [18]. The advantages and disadvantages of pressure-based coordinates are discussed below.

Horizontal discretization is the same as in HIRLAM, on a Arakawa C-grid. The advection scheme is currently Eulerian split-step, but a semi-implicit scheme is under development. A semi-Lagrangian advection scheme is also planned.

The lateral boundary conditions are treated as in Lokal-Modell, namely with Davies absorbing layer using a Rayleigh friction term in the governing equations. This is identical to HIRLAM because boundary relaxation is the disrete equivalent to the Davies absorbing layer.

The Tartu model uses the HIRLAM physics package, there are no special options for small scales or nonhydrostatic effects.

Since the model is in an early stage of development, there are no nesting capabilities.

Only very limited testing on workstations has been performed with the Tartu model, typically $50 \times 50 \times 31$ grids and resolution 10km. At this stage it is not relevant to compare the performance of the model with other the other models described above. The Tartu model is not parallelized as of now.

There is one other model which is under development, but which has reached a state where where it can be used for operational purposes, namely the MRI-NHM from Meteorological Research Institute in Japan. Comparative tests with the Lokal-Modell on idealized flows have been conducted recently, and show that MRI-NHM compares well with LM and even outperforms it on very high resolution, see [9].

4 Numerical methods development

For low and medium resolutions, semi-Lagrangian advection has proved to be more efficient than Eulerian schemes because the effective Courant number one can use by far outweigh the extra computations involved in interpolation etc. For high resolution models, this advantage is not so obvious any longer since there are different aspects of the models that start to affect the timestep, for example steep topography. There have been no comparisons between Eulerian and semi-Lagrangian schemes for realistic cases in terms of performance and quality of the results. For theoretical cases, there exist a few experiments between MC2 and LM, but no firm conclusions can be drawn yet.

On the other hand, semi-Lagrangian advection is now part of the standard fare in NWP, and the models using it, e.g. ALADIN and MC2 do not seem to be less efficient than other models. There is also a continuing development of semi-Lagrangian schemes, giving more accurate trajectory computations, see for example [23] and [24]. Thus it is by no means obvious what type of semi-Lagrangian scheme may turn out to be the most efficient for high resolution models.

When we start with the governing equations, the usual semi-implicit scheme (Eulerian or semi-Lagrange) consists of taking the advection part explicitly and then implicit treatment of sound waves and gravity waves. The implicit step is computed by solving a Helmholtz equation, and in order for this to be linear some linearization has to be performed.
Consider the quasilinear first-order system:

$$\frac{du}{dt} + \sum_{i=1}^{d} A_i(u) u_{x_i} + B(u) = F, \tag{4.1}$$

where u represents the vector of unknowns, and F the forcings and the diffusion. Linearization amounts to writing $u = u^* + u'$, where u^* is a reference value and u' the perturbation. If u^* is constant and A_i is linear, then we can write for the second term:

$$\sum_{i=1}^{d} \left(A_i(u^*) + A_i(u')\right) u'_{x_i},$$

which can be simplified if $\|A_i(u')\| \ll \|A_i(u^*)\|$, to:

$$\sum_{i=1}^{d} A_i(u^*) u'_{x_i}, \tag{4.2}$$

which is the form normally used. In some models the term $A_i(u')u'_{x_i}$ is taken at the previous timestep and used as a part of the forcing term. But in all the models A_i is not linear and the assumption $\|A_i(u')\| \ll \|A_i(u^*)\|$ depends of course on the solution and is not obvious in all cases. Hence there are limitations to the linearization approach that should be taken care of. For example in ALADIN, the straightforward linearization has proved to be unstable (Radmila Bubnova, personal communication). See also the literature for the discussions on the choice of reference states.

The time discretization schemes in the first nonhydrostatic models were split-explicit, i.e. using an explicit scheme to treat the acoustic and gravity waves, but with a smaller timestep than the advective timestep. The schemes in todays models are not only split-explicit, for example MM5 is using split-explicit, while MC2 and ALADIN uses semi-implicit. The original scheme in LM was split-explicit, but recently semi-implicit has been introduced, see [6]. Preliminary comparisons show that the semi-implicit scheme is competitive in terms of performance, and the two schemes give almost the same result on some theoretical cases as well as real cases. The new dynamics in the Unified Model also has a semi-implicit scheme, and testing on real cases is under way.

So the trend is to use semi-implicit schemes. However, semi-implicit schemes for nonhydrostatic models on small scales are not the same as for hydrostatic models on larger scales. In mesoscale models the pressure gradient term in the momentum equations are split into the horizontal part, taken, implicitly, and the vertical part taken explicitly. This simplification gives a symmetric positive definite (discrete) Helmholtz operator, and we can use very efficient solvers for such linear systems. However, this simplification can lead to significant errors in the presence of steep terrain, see e.g. [11] and [8]. Taking the full 3D pressure gradient term implicitly, results in a nonsymmetric Helmholtz equation (still elliptic). Since the linear systems involved are so large, only iterative solvers have been seriously considered. It is clear that solving a nonsymmetric linear system requires more resources than solving a symmetric positive definite one, but the developments in solvers for nonsymmetric systems has made this difference in computation less significant. For example, the GMRES family of methods, see [25], and the TFQMR family are powerful iterative solvers for nonsymmetric systems. As well known, the solvers will not work efficiently unless we can construct an efficient preconditioner. The field of constructing efficient preconditioners is still a very active one. In the current models, preconditioners based on classical iteration schemes (Jacobi, SOR, ADI) are used, but of course more sophisticated techniques like multigrid can be used.

One aspect that dominates the development of preconditioners is the parallelism required. There is no gain in a sophisticated preconditioner unless it can be run (very) efficiently in parallel since for large grids at high resolutions the Helmholtz solver can easily take 40-50 % of the time. Another aspect of the semi-implicit method was mentioned under the description of ALADIN, namely that one needs to take the full 3D divergence operator implicitly, see [13, Eqn.(20)]. This means that the Helmholtz equation will be nonlinear, but the weak nonlinearity will allow for a simple fixed-point outer iteration in the solution process. The appearance of a weakly nonlinear Helmholtz equation is not new, it has been discussed (and solved) by Staniforth and co-workers at RPN.

The operator splitting and the linearization discussed above will obviously affect the Helmholtz solver since the splitting directly determines the structure of the Helmholtz operator. In any case we can use efficient solvers for nonsymmetric linear systems, provided we can construct efficient preconditioners, and simple iteration schemes like fixed-point iteration since the nonlinearities that may arise is of 'logarithmic type'. The development of efficient, parallel linear equation solvers and preconditioner will certainly continue, and will be important for the practical use of the semi-implicit schemes.

The boundary conditions issue has been discussed extensively in the field of NWP, many times without looking at what is going on in related fields such as CFD. Note that the governing equations for most nonhydrostatic models is a quasi-linear hyperbolic system if we don't take diffusion terms into account. This structure makes it easier to use more rigorous methods for constructing boundary conditions, compared to the hydrostatic system which is rather special and harder to work with. Practically all the models use some form of boundary zone where the prognostic variables are relaxed towards some external field provided for example by a global model. This is done by direct relaxation:

$$\psi = \alpha(x)\psi^B + (1 - \alpha(x))\psi^I, \tag{4.3}$$

where ψ^B is the boundary value, ψ^I the value computed from the interior of the domain and ψ the relaxed value. The relaxation factor α varies from 0 to 1 going from the interior to the boundary. In the continuous case the relaxation method may be formulated as a Rayleigh friction term in the prognostic equations:

$$\frac{\partial \psi}{\partial t} = F_\psi - K(x)(\psi - \psi^B), \tag{4.4}$$

where F_ψ is the ordinary tendency in ψ, and $K(x)$ is the space-dependent friction coefficient.

Many models do not distinguish between outflow and inflow boundaries, the relaxation is done in the same way everywhere. In recent developments, for example in MC2 and LM, see [4] and [6], transparent boundary conditions have been implemented by imposing values of the prognostic variables at the inflow points and extrapolating them from the interior at the outflow points. The boundary condition for the Helmholtz equation is then derived from the discrete operators. This is only a partial solution, one should impose values for the incoming characteristic variables.

A lot of work has been done in the field of CFD trying to construct efficient transparent boundary conditions, and some of this work should be considered when developing new boundary conditions for NWP. In computational electromagnetics a method called Perfect Matching Layer (PML) is now in extensive use, see e.g. [27]. This method an elaboration of Rayleigh friction methods above, and it may be worthwhile to use some of the ideas from PML when constructing relaxation schemes.

The most rigorous method for constructing transparent boundary conditions is the Dirichlet-to-Neumann map, (Neumann map). The idea here is that in theory the governing equations should be solved everywhere, the boundary is something we don't want to see. So we take the PDE system representing the governing equations and write it as a Neumann boundary condition:

$$\frac{\partial u}{\partial n} = N_\Omega(u), \tag{4.5}$$

where N is the Neumann map, a pseudodifferential operator. This operator is not practical for computing, and all the work on this is concentrated on finding good approximations. We expect that even if the rigorous approach is rather complicated to work with in NWP, further development of boundary conditions will benefit from using it.

With respect to performance on parallel computers, there is no compromise, all the models must run very efficiently on the parallel platforms in the operational centers. The development of numerical methods for NWP must be based on this requirement. Very good performance will presumably also in the future not come easily by using some standard tools, some "parallel hacking" has to be done to achieve the best results. This statement is based on practical experience, and also on broken promises by computer manufacturers and software developers.

Most of the current nonhydrostatic models of today scale very well with the number of processors, even if relatively straightforward data partitioning is used. Domain decomposition techniques, for example subdomain solvers and alternating Schwarz methods [26], have not been used in NWP yet, but the Helmholtz equation in the models is a good candidate for such techniques.

Most limited area models use a regular grid in the horizontal under some map projection, and some terrain-following coordinate in the vertical. Méteo-France is using variable grid (Paris-centered) for ARPEGE, and the Canadian GEM model is also using a variable grid, but the principle for refinement is quite different.

Some experiments have been performed with variable grids in slightly different versions of the current HIRLAM model, and the results are encouraging, but it is difficult to tell if there is any accuracy and efficiency gain compared to regular grids.

For high resolution models the need for variable grids are probably less because the grids will cover a relatively small area where we have sufficient resolution to resolve say steep topography. However, this issue is related to boundary conditions: If one does not use efficient boundary conditions, there is a need to have the boundary far away from the region of interest. Adaptive grids have been used in research, but have not migrated into operational models yet.

Except for the German GME global model which uses a icosahedral/ tetrahedral grid, only rectangular horizontal grids have been used for grid point models. Spectral models have their grids following the nodes in quadrature formulas, but the only spectral nonhydrostatic model described above, ALADIN, is using Fourier collocation methods, and hence a regular grid in spectral space. There is currently no evidence that variable grids are always superior to regular grids for high resolution models, but efficiency (using fewer gridpoints in parts of the domain) may render variable grids an interesting option. The related question of gridpoint vs. spectral methods will not be settled in the near future, there are advantages and disadvantages with both alternatives which will be discussed briefly below.

The parallel aspects of the models and the development of efficient numerical techniques is connected to many computer science aspects of the models. Some of them are mentioned below.

The field of NWP is Fortran oriented. For efficiency and compatibility reasons Fortran will probably always be there. Today's Fortran-90 compilers have reached a reasonably high level of quality and exist on almost all platforms. This should imply that one should use Fortran-90. In the field of parallelism, High Performance Fortran (HPF) is a nice concept but at least today not able to produce good performance code without a lot of work with the code. As a consequence of this, the message passing programming model using MPI (or relatives) will probably be the way we develop parallel code also for the next generation of NWP models. Today, almost all models are using MPI.

Structuring of the model code is always important. With nonhydrostatic models there is an active development of new numerical methods, so it is advantageous to construct the model code such that modules, for example elliptic solvers, can be exchanged easily. Another example is parts of the physics package. This is not the place to discuss how model codes should be designed, but care should be taken to allow for easy update and change of parts of the code.

To some extent the development of numerical methods will have to take into account the computer architecture for the platforms we use. In theory we could design high performance parallel codes for any platform, but to get the really good performance some architecture-dependent hacks will be needed. Today it seems that the number of architectures for high performance parallel computing is limited, so it should not be a difficult task.

As mentioned above, we will not discuss the physical parameterization for high resolution models here. Some of the points that need to be discussed are:

- More prognostic variables, for example rainwater and ice
- Better (more detailed) description of processes that cannot be treated explicitly
- What can be omitted from the physics when going to higher resolution
- Better numerical schemes, and parallelism also in the vertical

The limited practical experience with parameterizations in high resolution models seems to indicate that having good physics will be at least as important as it is for today's models.

Data assimilation for high resolution models is a grand challenge that probably will determine if high resolution models will be useful for operational NWP. There are only a few cases where just increasing the horizontal or vertical resolution without doing anything with analysis schemes and physics give better forecasting results. Data assimilation and modeling are intrinsically linked in the 4DVAR analysis schemes, so the development of a model might just be seen as a part of the data assimilation. On the other hand there is no obvious requirement to use exactly the same model in a 4DVAR scheme and in the forecasting model.
Some topics to be discussed are:

- Resolution in the 'forward model' in 4DVAR compared to the prognosis model
- How to use the increasing number of observations available
- The performance and practical use of 4DVAR for limited area models
- Analysis schemes other than 3DVAR and 4DVAR

It would be valuable to have additional reports on physics and data assimilation for high resolution models, so that the HIRLAM community have a basis for the discussion of how the future HIRLAM should look.

5 Recommendations for HIRLAM

In this section we will try to sketch some alternatives for a nonhydrostatic model for the HIRLAM group. The material discussed in the previous sections forms a basis for the discussions. Some conclusions can be drawn from the experiences with the models so far and the development of numerical methods. However, most of the choices that have to be made will be based on personal opinions and assumptions. What I am trying to do is not to present clearly distinct alternative choices for constructing a nonhydrostatic model, but rather a list of issues that have to be taken into account, accompanied by a discussion. It is time to reiterate what was stated in the introduction: the viewpoints presented here are those of the author, and not necessarily those of DNMI or the HIRLAM management. There will always be different opinions on how to build a nonhydrostatic model for operational purposes, and it is not the intention to criticize certain models and argue strongly with scientists in the field. However, it is unavoidable to comment on certain details of the models, and therefore their authors.

5.1 Model structure

By model structure we mean how the nonhydrostatic model is a part of a model set and the relations between the members of this set. Two alternatives are immediate, either building a new nonhydrostatic model and throw away the current one, or use the new model as a part of a model complex, for example by nesting it into the current model. We have seen both approaches in practice, the UM is an example of the first alternative, while LM is an example of the second alternative. From the experience with the nonhydrostatic models so far, there seems to be few difficulties in using nonhydrostatic models on a grid size of 50 km, see e.g. [2]. So within the resolution range of the current national HIRLAM variants one could run one model. Some models, for example MC2 and ALADIN, have a switch to run in hydrostatic mode. Enabling the hydrostatic switch does not imply saving a lot of computer time and space, so it is more useful to compare results rather than a model system being efficient in both nonhydrostatic and hydrostatic mode.

Note that developing a new model to replace the current one does not imply that all the efforts gone into the development of the current model are wasted. The physics package can hopefully be used almost as before, and if one can use the Laprise approach in the formulation of a nonhydrostatic model, some of the model can be used as it is.

5.2 Model development

Building a nonhydrostatic model is a very resource-demanding task, and hardly any model has been built entirely from scratch. The development of LM is a good example here. This model development started out with MM5, but has evolved into something different, the vertical coordinate has been changed to terrain-following z-coordinate, rotated lat-lon grids have been introduced, and quite a few other minor changes. The development of LM was completed in a relatively short period of time, so this approach would also be feasible for HIRLAM where the development will be distributed (in space) and where it is difficult to mobilize a large work force for a concentrated effort.

A different approach was taken when UKMO developed the new dynamics of the UM. The new dynamics is a radical change to the previous version of the model, and the risk of having to deal with too many problems simultaneously is clearly there. To complete such a task successfully one has to command the work force very efficiently, and that is difficult within HIRLAM.

The MC2 model was developed from the research model of André Robert, and the first version of the model was completed in 2 years. However, the model has been improved considerably since the first version, and the total development time until now is about 4 years with a work force varying from 5-10 full-time scientists. This does not include development of most of the physics package. The MC2 approach, i.e. starting from a research model, could also be possible for HIRLAM, but will of course require more resources than by starting from a complete model.

5.3 Advection schemes

We have seen from the brief descriptions of the models above that semi-Lagrangian advection is now used in MC2, ALADIN and the new dynamics of the UM. The LM and the MM5 use Eulerian advection.

There are in fact no realistic comparisons in terms of performance between the two advection schemes. To do that one has to compare different models, and not only the difference in the advection. The MC2 and the LM has been compared on theoretical cases, but not on the same computer platform, so the results are not directly useful. From our own numerical experiments it seems that the performance of MC2 and MM5 are about the same, but this is again difficult to interpret because the timestep has not been optimized for the two models, and the resulting fields have only been compared against each other, not properly verified.

As mentioned above the advantage of taking longer timestep for semi-Lagrangian advection compared to Eulerian is not so obvious for high resolution models. On the other hand, semi-Lagrangian advection does not seem to be significantly worse than Eulerian in any cases, so if a nonhydrostatic HIRLAM is going to be used on a range of resolutions, semi-Lagrangian advection seems to be a reasonable choice. With respect to parallelism, numerical experience have shown that semi-Lagrangian schemes are quite good, but it seems that trajectory computations can still be improved.

5.4 Horizontal discretization

In the models described above, the horizontal discretization is either finite differences or spectral. The spectral projection in the models is collocation. Note that we limit our discussion to limited area models, spectral methods have their well known advantages for global models. Finite element methods have been used in models developed by RPN in Canada, but have not been used in the models in Europe. Some of the finite difference methods used are in fact similar to finite volume methods.

There has been so much work on numerical methods for both FD and spectral discretization, that it is impossible to say which is the better only based on the state of the art in numerical methods for NWP. Note that in this context we assume that spectral discretization means Fourier discretizations. Non-periodic spectral bases such as Legendre and Chebyshev seem to be less suitable for NWP because of the more strict timestep restrictions than for Fourier methods. The price to pay with Fourier methods is of course the periodization of the domain, which can cost quite a few gridpoints. Note that there is a development going on in trying to use Fourier methods for non-periodic functions, by reconstruction of spectral components of discontinuous functions, see e g. [29].

The choice between FD or spectral is usually made only based on personal preference. Of course there are advantages and disadvantages of both, for example solving a linear Helmholtz equation in spectral space is trivial and the cost is only the transforms. On the other hand, experiments have shown that the Helmholtz equation in ALADIN have to be nonlinear, and that partly destroys the advantage since the outer iteration must be done in physical space.

The development of numerical methods will mean that at one instant one alternative will seem superior, some time later the situation may be different. This is not to say that there are differences between the two horizontal discretizations, but the differences may show up in apparently less obvious aspects, such as coupling to physics and interaction with variational data assimilation schemes. As of today, it is impossible to say, at least on a scientific basis, what discretization comes out as the most efficient one.

5.5 Pressure coordinate models

The Tartu model described above is using pure pressure as the vertical coordinate. Model formulations with pressure are well known from the literature, and have interesting properties. However, such models have not been used in other than experimental models, so one should be careful when claiming that they are better than other model formulations in operational use. It is difficult to predict what problems one may encounter in the discrete formulation. For example, the problem encountered with the Helmholtz equation in ALADIN, which is nonlinear because the complete velocity divergence needs to be taken implicitly, is not seen in the Laprise formulation. The Tartu model is not yet semi-implicit, so it is not ready for comparison with the other model formulations. There is no semi-Lagrangian option either, and without coupling to a full physics package realistic testing of the numerical formulation is not possible.

To me it seems a bit risky to base the development of a nonhydrostatic HIRLAM model solely on the Tartu model. This is because it is still in an early stage of development and consequently it will take some time before one can compare it with other models in operational use. It is to be emphasized that this viewpoint is not taken because I think the model will be bad, but the properties of the numerical formulation of the model have not been presented yet, and from experience quite a lot of testing needs to be done before one can claim that the model is "stable".

5.6 The Laprise approach

The compressible Euler equations has been formulated in the hydrostatic pressure-based η-coordinate by Laprise, see [14]. There are some advantages of this approach:

- The resulting equations are a "superset" of the hydrostatic equations.
- One can easily switch between nonhydrostatic and hydrostatic mode
- The diagnostic relations for $\frac{dp_s}{dt}$, ω and $\dot{\eta}$ are the same as in HIRLAM.

More details on the Laprise formulation in the context of (today's) HIRLAM is given in Appendix A. Note that the Laprise formulation only covers the continuous equations, numerical aspects like the horizontal discretization and semi-implicit scheme are not covered. Boundary conditions were not discussed in [14] either, but it should not be more difficult to derive transparent boundary conditions with η-coordinates compared to z-based coordinates. Radiation type upper boundary conditions are derived for η-coordinates in [28].

As mentioned above, ALADIN is using the Laprise approach with some adaptations, see [13]. The implementation of the Laprise formulation in ALADIN seems to contain some problems, and it is worthwhile to quote Laprise (personal communication, 1998): "If one should use the Laprise formulation, it would be best not to use the ALADIN formulation as a basis for a nonhydrostatic HIRLAM model, because there are details we are not satisfied with. The operator splitting and the linearization are two of those details. For a new model I would suggest a careful rethinking of the numerical formulation." Note that the need for a nonlinear Helmholtz equation mentioned above is not a weakness of the Laprise formulation itself, but a consequence of the possible operator splittings. There exist quite a few papers in the mathematical and numerical literature which could be useful for a broader discussion on operator splittings.

Apart from ALADIN, two other nonhydrostatic models are using the Laprise formulation, namely NCEP's meso-scale ETA model and the Canadian GEM model. These two models are usually run on large domains (the GEM is global), and it is difficult to put them in the class of high resolution models.

Practical issues such as coupling between dynamics and physics has of course been addressed in ALADIN, but Laprise has pointed out that there are still problems, for example certain stationary forcings seems to produce results of lower accuracy than expected.

6 Conclusion

In this report I have tried to come up with recommendations for the future nonhydrostatic HIRLAM, based on experience with the existing models so far and some scientific background. The recommendations may seem vague and hence not very useful. This has not been the intention, but several aspects, some of them personal preferences and pragmatic viewpoints, make it difficult to reach firm conclusions:

- Aspects of the models, such as coordinate systems, semi-Lagrangian vs. Eulerian advection etc. do not present themselves as alternatives that can be clearly ranked.
- The numerical methods are in a phase of fairly rapid development.
- The experience with models so far is different from model to model and does not give a coherent picture.
- Some people are more eager to present their ideas and results than others.

It is my hope that this report can be a starting point for the discussion on what to do and how to construct a nonhydrostatic HIRLAM. I hope there will be many many discussions on the topic in the future, and hopefully one can come out with something that suits the HIRLAM community well.

Acknowledgment
I would like to thank Terry Davies, Jean-Francois Geleyn, Michel Desgagne, Steve Thomas, Jürgen Steppeler, Rene Laprise and Kazio Saito who has sent me information on the different models, and answered my questions. Without this information, this report could not be written.

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A Using the Laprise approach in HIRLAM

A.1 Introduction

In 1992 Laprise [14] introduced a new approach for the Euler equations by using the same vertical coordinate system, the so-called η coordinates, commonly used in hydrostatic models. This coordinate system is used at ECMWF and in the HIRLAM model. The formulation by Laprise shows that many of the advantages of the pressure-related coordinates well known from hydrostatic models could be carried over to the nonhydrostatic case making it easier to construct a common model framework for the hydrostatic and nonhydrostatic cases.

The ALADIN model at Méteo France [13] is an implementation of the Laprise approach with minor modifications. The results from ALADIN are still preliminary in the context of operational use. Note that Laprise [14] only describes the dynamics part for a dry atmosphere, and it may well be the case that there will be problems in using say the current HIRLAM physics package in a nonhydrostatic model, at least on finer scales.

There are different vertical coordinates used in nonhydrostatic models, all with some advantages and disadvantages. The MM5 model [7] is using a σ coordinate, while the Canadian MC2 model is using the z coordinate with the Gal-Chen transform, see [2]. The HIRLAM-based nonhydrostatic model under development by R.Room and colleagues in Tartu is using a pure pressure coordinate.

This appendix will describe how the Laprise approach can be incorporated into the HIRLAM dynamics part. We will assume that the reader knows the governing equations for HIRLAM, so we will only refer to the HIRLAM Documentation Manual [15] for details and further information.

A.2 Vertical coordinate and governing equations

In this section we will describe briefly the η coordinate and formulate the governing equations as given in [14]. The relation to the HIRLAM governing equations will be detailed in the next section.

The hybrid hydrostatic pressure coordinate η is given by the following relation:

$$\pi(x, y, \eta, t) = A(\eta) + B(\eta)\pi_s(x, y, t). \tag{A.1}$$

In this relation $\pi(x, y, \eta, t)$ is the hydrostatic pressure, $\pi_s(x, y, t)$ is the hydrostatic surface pressure and $A(\eta)$ and $B(\eta)$ are smooth functions. A and B can be arbitrary, but for the lowest η surface to follow the topography, η must be constant along the surface $\pi = \pi_s$, hence $A(\eta_s) = 0$, $B(\eta_s) = 1$, where η_s is normally 1. Moreover, if the upper boundary is to correspond to a η surface then $A(\eta_T) = \pi_T \geq 0$, $B(\eta_T) = 0$.

The governing equations for a dry atmosphere are as follows:

$$\frac{d}{dt}\mathbf{V} + RT\nabla_\eta \ln p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla_\eta \phi = \mathbf{F}, \tag{A.2}$$

$$\gamma\frac{dw}{dt} + g\left(1 - \frac{1}{m}\frac{\partial p}{\partial \eta}\right) = \gamma F_z, \tag{A.3}$$

$$c_P\frac{dT}{dt} - \frac{RT}{p}\frac{dp}{dt} = Q, \tag{A.4}$$

$$\frac{d\ln p}{dt} + \frac{c_P}{c_V}D_3 = \frac{Q}{c_V T}, \tag{A.5}$$

$$\left(\frac{\partial m}{\partial t}\right)_\eta + \nabla_\eta \cdot (m\mathbf{V}) + \frac{\partial}{\partial \eta}(\dot{\eta} m) = 0, \tag{A.6}$$

$$\alpha = \frac{RT}{p}, \tag{A.7}$$

$$\phi = \phi_s + \int_\eta^{\eta_s} \alpha \frac{\partial \pi}{\partial \eta'} d\eta', \tag{A.8}$$

where

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\eta + \mathbf{V} \cdot \nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta}, \tag{A.9}$$

$$D_3 = \nabla_\eta \cdot \mathbf{V} + \frac{1}{m}\rho \nabla_\eta \phi \cdot \frac{\partial \mathbf{V}}{\partial \eta} - \frac{1}{m}\rho g \frac{\partial w}{\partial \eta}. \tag{A.10}$$

In these equations $\mathbf{V}$ is the horizontal velocity vector, w is the vertical velocity, T is the temperature, p is the pressure, π is the hydrostatic pressure, α the specific volume, ϕ the geopotential height, c_P and c_V the heat capacities at constant pressure and constant volume respectively, R the gas constant, Q the heat source term, $\mathbf{F}$ and F_z forces (other than gravity) in the momentum equations and $\dot{\eta} = \frac{d\eta}{dt}$. We have also used the abbreviation $m = \frac{\partial \pi}{\partial \eta}$. The parameter γ determines the type of model. If $\gamma = 1$ we have the nonhydrostatic model, and if $\gamma = 0$ the pressure becomes the hydrostatic pressure and (A.3) reduces to the equation for hydrostatic balance. In this case the pressure equation (A.5) becomes a diagnostic relation for w.

Several forms of the mass continuity equation can be used. We may for example use ϕ as a prognostic variable and in place of (A.5) use the equation

$$\frac{d\phi}{dt} - wg = 0, \tag{A.11}$$

and find α and p from the diagnostic relations

$$p = \frac{RT}{\alpha}, \qquad \alpha + \frac{\partial \phi}{\partial \eta}\frac{1}{m} = 0. \tag{A.12}$$

In any case it is wise to use the pressure deviation from the hydrostatic pressure as a variable instead of the pressure itself because of the second term in (A.3). This is done in ALADIN, where the pressure deviation is scaled with the reference hydrostatic pressure. Let

$$\mathcal{P} = \frac{p - \pi}{\pi^*}$$

be the scaled pressure deviation, and π^* the reference hydrostatic pressure. Since we have

$$\frac{1}{m}\frac{\partial p}{\partial \eta} = 1 + \frac{1}{m}\frac{\partial}{\partial \eta}(\pi^* \mathcal{P}),$$

we get the vertical momentum equation:

$$\gamma \frac{dw}{dt} - \frac{g}{m}\frac{\partial}{\partial \eta}(\pi^* \mathcal{P}) = \gamma F_z. \tag{A.13}$$

The horizontal momentum equation is obtained by transforming $\nabla_\eta p$:

$$\nabla_\eta p = \pi^* \nabla_\eta \mathcal{P} + \nabla_\eta \pi.$$

Using the same transformation as for the vertical momentum equation, we obtain:

$$\frac{d\mathbf{V}}{dt}+\frac{RT}{\pi^*\mathcal{P}+\pi}\left(\pi^*\nabla_\eta\mathcal{P}+\nabla_\eta\pi\right)+\left(\frac{1}{m}\frac{\partial}{\partial\eta}(\pi^*\mathcal{P})+1\right)\nabla_\eta\phi=\mathbf{F}. \tag{A.14}$$

Note that $\nabla_\eta\pi^*=0$ and $\nabla_\eta\pi=B(\eta)\nabla\pi_s$.
Consider now the pressure equation. The total derivative of the pressure becomes:

$$\frac{dp}{dt}=\frac{d}{dt}(\pi^*\mathcal{P})+\frac{d\pi}{dt}=\pi^*\frac{d\mathcal{P}}{dt}+\mathcal{P}\dot\eta\frac{\partial\pi^*}{\partial\eta}+\dot\pi,$$

where $\dot\pi=\frac{d\pi}{dt}$. Hence by division by π^* we obtain:

$$\frac{d\mathcal{P}}{dt}+\frac{\mathcal{P}\dot\eta}{\pi^*}\frac{\partial\pi^*}{\partial\eta}+\frac{\dot\pi}{\pi^*}+\frac{c_P}{c_V\pi^*}(\pi^*\mathcal{P}+\pi)D_3=\frac{Q(\pi^*\mathcal{P}+\pi)}{\pi^*c_VT}. \tag{A.15}$$

A.3 HIRLAM and the Laprise system

The Laprise system of equations contains two more prognostic variables than today's hydrostatic HIRLAM, namely the (true) pressure and the (true) vertical velocity. The hydrostatic pressure has to be used because of the coordinate system and has also definite advantages for deriving diagnostic relations, see below. The (true) vertical velocity is also required in nonhydrostatic models based on the Euler equations. In ALADIN, the pressure deviation is used as the prognostic variable, but note that the geopotential could still be used as prognostic since the pressure deviation could be found from the modified diagnostic relation

$$\pi+\pi^*\mathcal{P}=\frac{RT}{\alpha}. \tag{A.16}$$

Some of the diagnostic relations used in HIRLAM are identical to those used in the Laprise system and ALADIN. For example, integration of (A.6) through the depth of the atmosphere gives the surface pressure tendency equation

$$\frac{d\pi_s}{dt}+\nabla\cdot\int_0^1 m\mathbf{V}d\eta=0, \tag{A.17}$$

which is identical to (2.1.1.13) in [15]. Similarly integrating (A.6) from the top of the atmosphere to the current level gives the pseudovertical velocity in η coordinates:

$$\begin{aligned} m\dot\eta &= B(\eta)\int_0^1\nabla\cdot(m\mathbf{V})d\eta-\int_0^\eta\nabla\cdot(m\mathbf{V})d\eta \\ &= \frac{d\pi}{dt}-\frac{d\pi_s}{dt}+\int_\eta^1\nabla\cdot(m\mathbf{V})d\eta. \end{aligned} \tag{A.18}$$

The last line follows by using (A.17) and the relation $\frac{d\pi}{dt}=B(\eta)\frac{d\pi_s}{dt}$ which comes from the coordinate definition. The result is in practice (2.1.1.15) in [15]. The total derivative of the hydrostatic pressure $\dot\pi$ usually denoted by ω can now be found from (A.17) and (A.18):

$$\begin{aligned} \dot\pi=\omega &= \mathbf{V}\cdot\nabla\pi-\int_0^\eta\nabla\cdot(m\mathbf{V})d\eta \\ &= \frac{d\pi_s}{dt}+\int_\eta^1\nabla\cdot(m\mathbf{V})d\eta+\mathbf{V}\cdot\nabla\pi, \end{aligned} \tag{A.19}$$

and this is the same as (2.1.1.14) in [15]. also note that the the derivative of (A.8) w.r.t. η,

$$\frac{\partial \phi}{\partial \eta} = -m\frac{RT}{p} = -\frac{\partial \pi}{\partial \eta}\frac{RT}{p}, \tag{A.20}$$

is identical to the HIRLAM hydrostatic relation, (2.1.1.10) in [15]. Laprise discusses only the governing equations for a dry atmosphere, so in the HIRLAM context we have to augment the system with continuity equations for humidity q and cloud water q_l:

$$\frac{dq}{dt} = P_q + K_q \tag{A.21a}$$

$$\frac{dq_l}{dt} = P_{q_l}. \tag{A.21b}$$

Here P_q and P_{q_l} denotes the tendencies from the physics, and K_q the contribution from the horizontal diffusion. Inclusion of moisture introduces changes in the momentum and energy equations, and in the equation of state. However, these changes do not change the relation between the diagnostic relations in HIRLAM and the Laprise formulation.